PARTITIONS OF A POLYTOPE AND MAPPINGS OF A POINT SET TO FACETS

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Abstract. Three theorems of this paper generalize previous results of the author on conjectures of A. Bezdek and V. V. Proizvolov. They show the existence of mappings from a given point set to the set of facets of a given polytope that satistfy some special conditions.

Developing the same technique, some results on convex polytope partitions are presented, two of them dealing with partitions with prescribed measures of parts. Then we prove a corollary on existence of a possibly non-convex polytope with given set of vertices, containing given points in its interior.

We also consider problems of the following type: find an assignment of vectors from a given set to the parts of a given convex partition of $\mathbb{R}^n$ so that the shifts of the parts by their corresponding vectors either do not intersect by interior points or cover $\mathbb{R}^n$.

1. Introduction

Theorems 1–3 generalize Theorems 2, 3, and 4 of paper [1]. They show the existence of mapping f of a given finite set in $\mathbb{R}^n$ to the set of facets of a given polytope such that some conditions hold: for example, the convex hulls $\operatorname{conv} F \cup f^{-1}(F)$ for different facets F do not intersect pairwise by interior points.

Some of the results in [1] may be considered as a case of Theorems 1–3 when the number of points to be mapped equals the number of facets.

These theorems also lead to Corollary 2 about convex partitions of $\mathbb{R}^n$ that generalizes the corollaries from [1]. Corollary 2 shows that for partition of a special type, defined in this paper, one can find an assignment of vectors from a given set to the parts of this convex partition of $\mathbb{R}^n$ so that the shifts of the parts by their corresponding vectors either do not intersect by interior points or cover $\mathbb{R}^n$.

Theorems 4 and 5 generalize Theorem 4 from [2] to the case of $\mathbb{R}^n$. They deal with the following problem: find a partition of a given polytope such that its parts have given measures and every part contains a prescribed facet of the polytope.

Now we introduce some notation. Let $V \subseteq \mathbb{R}^n$, then

(1) $\operatorname{conv} V$, $\operatorname{lin} V$, $\operatorname{aff} V$ denote convex, linear and affine hull of V;
(2) $\operatorname{int} V$, $\operatorname{rint} V$, $\operatorname{cl} V$, $\operatorname{bd} V$, $|V|$ denote interior, relative interior (if V is convex), closure, boundary, number of elements of V;
(3) $\dim_l V$, $\dim_a V$, $\operatorname{codim} V$ denote linear dimension, affine dimension, codimension of V;
(4) $I_m = \{1, 2, \dots m\}$ denotes the set of indices and S^m denote the group of permutations of I_m;

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(5) $\operatorname{dist}(x,y)$, $\operatorname{dist}(X,Y)$ denote the distance between points x and y or subsets X and Y of some metric space;
(6) For $A, B \subseteq \mathbb{R}^n$ denote

$$A + B = \{a + b : a \in A,\ b \in B\} \quad A \overset{*}{-} B = \{x : x + B \subseteq A\}$$

the Minkowski sum and the geometric difference of A and B.

Let L_1^n be a space of affine functions (polynomials of degree 1) on $\mathbb{R}^n$. Suppose $f \in L_1^n$; by definition, put

(1) $H(f) = \{x \in \mathbb{R}^n : f(x) = 0\}$ — the hyperplane;
(2) $H^-(f) = \{x \in \mathbb{R}^n : f(x) < 0\}$ — the open halfspace;
(3) $H^+(f) = \{x \in \mathbb{R}^n : f(x) \geq 0\}$ — the closed halfspace.

Recall that a *polyhedral set* is an intersection of a family of halfspaces

$$X = \bigcap_{i \in I_m} H^+(f_i).$$

Suppose X is a polyhedral set, denote $F_i = H(f_i) \cap X$. We will assume that $\operatorname{codim} F_i = 1$ for all $i \in I_m$ and define the *facets* of X to be the sets $\{F_i\}_{i \in I_m}$.

Also, put $H_i = H^-(f_i)$ for all $i \in I_m$.

If a polyhedral set is bounded, then it is called a polytope.

Theorem 1. *Let X be a polytope in $\mathbb{R}^n$, let $\{F_i\}_{i \in I_m}$ be its facets. Let $A \subset \mathbb{R}^n$, $l = |A|$, and let $\{l_i\}_{i \in I_m}$ be a set of positive integers such that $\sum_{i \in I_m} l_i = l$.*

Then there exists a partition of A into A_i ($i \in I_m$) satisfying the following conditions:

1) *$|A_i| = l_i$ for any $i \in I_m$;*
2) *let $C_i = \cap_{a \in A_i} \operatorname{conv}(a \cup F_i)$, then sets $\{C_i\}_{i \in I_m}$ cover X, in other words,*

$$X \subseteq \bigcup_{i \in I_m} C_i.$$

Theorem 1 is formulated for a bounded polytope, so it is natural to ask what can be done for an unbounded polyhedral set. This theorem is true for a given set A and a polytope. Then we can apply a projective transformation that maps one facet to a subset of the hyperplane at infinity; note that any polyhedral set that does not contain a straight line can be obtained this way.

Theorem 1 will be true for such an unbounded polyhedral set X if we consider the facet at infinity F_m along with $m - 1$ normal facets, denote

$$I = \{x \in \mathbb{R}^n : \forall \lambda > 0 \quad \lambda x \in X - X\},$$

and put $C_m = \cap_{a \in A_i}(a + I)$ (and C_i for $i = 1, \dots, m-1$ as in Theorem 1).

See section 6 for Corollary 2, which may be considered as a generalized form of Theorems 1 and 2 applied for polyhedral sets with “all facets at infinity”.

Theorem 2. *Let X be a polyhedral set in $\mathbb{R}^n$, let $\{F_i\}_{i \in I_m}$ be its facets. Let $A \subset X$, $l = |A|$, and let $\{l_i\}_{i \in I_m}$ be a set of positive integers such that $\sum_{i \in I_m} l_i = l$.*

Then there exists a partition of A into A_i ($i \in I_m$) satisfying the following conditions:

1) *$|A_i| = l_i$ for any $i \in I_m$;*

2) *let* $C_i = \operatorname{conv}(A_i \cup F_i)$, *then the sets* $\{C_i\}_{i \in I_m}$ *do not intersect pairwise by interior points, in other words, for all* $i \neq j \in I_m$

$$\operatorname{int} C_i \cap \operatorname{int} C_j = \emptyset.$$

Theorem 3. *Let* X *be a polyhedral set in* $\mathbb{R}^n$, *let* $\{F_i\}_{i \in I_m}$ *be its facets. Let* $A \subset \mathbb{R}^n \setminus X$, $l = |A|$, *and let* $\{l_i\}_{i \in I_m}$ *be a set of positive integers such that* $\sum_{i \in I_m} l_i = l$. *Suppose that* $|A \cap \bigcup_{i \in I} H_i| \geq \sum_{i \in I} l_i$ *holds for all* $I \subseteq I_m$.

Then there exists a partition of A *into* A_i $(i \in I_m)$ *satisfying the following conditions:*

1) $|A_i| = l_i$ *for any* $i \in I_m$;

2) *let* $C_i = \operatorname{conv}(A_i \cup F_i)$, *then the sets* $\{C_i\}_{i \in I_m}$ *do not intersect with* X *by interior points and do not intersect by interior point pairwise. In other words, for any* $i \in I_m$

$$\operatorname{int} C_i \cap \operatorname{int} X = \emptyset,$$

and for all $i \neq j \in I_m$

$$\operatorname{int} C_i \cap \operatorname{int} C_j = \emptyset.$$

Theorems 2, 3, and 4 from [1] may be obtained from the above theorems if $|A| = m$ and $l_i = 1$ for all $i \in I_m$.

The same method allows us to prove three other theorems. The following too theorems can be considered as continuous cases of Theorem 2. Here we have a measure instead of a finite point set.

Theorem 4. *Let* X *be a polytope in* $\mathbb{R}^n$, *let* $\{F_i\}_{i \in I_m}$ *be its facets. Let* μ *be a measure on* X, *continuous w.r.t the Hausdorff metric, and let* $\mu(X) = 1$. *Then for any set of positive numbers* $\{\mu_i\}_{i \in I_m}$ $(\sum_{i \in I_m} \mu_i = 1)$ *there exists a partition of* X *into convex sets* $\{A_i\}_{i \in I_m}$ *such that for all* $i \in I_m$

$$A_i \cap \operatorname{bd} X = F_i \quad \mu(A_i) = \mu_i.$$

We formulate a slightly different theorem that generalizes Theorem 4 from [2] to the case of $\mathbb{R}^n$:

Theorem 5. *Let* X *be a polytope in* $\mathbb{R}^n$, *let* $\{F_i\}_{i \in I_m}$ *be its facets. Let* μ *be a measure on* X, *continuous w.r.t the Hausdorff metric, and let* $\mu(X) = 1$. *Let* $\{\mu_i\}_{i \in I_m}$ *be a set of real numbers such that* $\sum_{i \in I_m} \mu_i = 1$ *and* $\mu_i \geq 0$ *for all* $i \in I_m$. *Denote* $I = \{i \in I_m : \mu_i > 0\}$, *then there exists a partition of* X *into convex sets* $\{A_i\}_{i \in I}$ *such that for all* $i \in I$

$$A_i \supseteq F_i \quad \mu(A_i) = \mu_i.$$

Theorem 6. *Let* X *be a polytope in* $\mathbb{R}^n$, *let* $\{F_i\}_{i \in I_m}$ *be its facets. Let* $V \subset \operatorname{int} X$ *be a finite set and let* $m \geq |V| + 1$. *Then there is a partition of* X *into convex sets* $\{A_i\}_{i \in I_m}$ *such that for all* $i \in I_m$

$$A_i \cap \operatorname{bd} X = F_i \quad V \cap \operatorname{int} A_i = \emptyset.$$

Remark. In fact, the partitions in these three theorems are *affine* partitions (see section 6).

Theorem 6 leads to the following corollary:

Corollary 1. *Let* W *and* V *be disjoint finite sets in* $\mathbb{R}^n$, *let their union be in general position. Let* $V \subset \operatorname{conv} W$, *let* $\operatorname{conv} W$ *have* m *facets, and let* $m \geq |V| + 1$. *Then there exists a possibly non-convex polytope* Y *with vertices* W *such that* $V \subset Y$.

Remark. In this theorem non-convex polytope is a bounded closed set $Y \subset \mathbb{R}^n$ such that $\operatorname{bd} Y$ is a polyhedral complex, homeomorphic to $(n-1)$-dimensional sphere.

2. Auxiliary assertions

Here we give some definitions and lemmas. The lemmas are taken from [1], so we do not give proofs here.

Definition. We say that the set V in a linear (affine) space L (A) is in *general position* and write $V \in LGP$ $(V \in AGP)$ if

$$\dim_l U = \min\{|U|,\ \dim_l L\} \quad (\dim_a U = \min\{|U|-1,\ \dim_a A\})$$

for any finite $U \subseteq V$.

Definition. Suppose $\mathcal{F}$ is a family of maps from the same finite set I to L (A), then we say that the map $\lambda \in \mathcal{F}$ is in *general position* and write $\lambda \in LGP(\mathcal{F})$ $(\lambda \in AGP(\mathcal{F}))$ if

$$\dim \lambda(U) = \max_{\lambda' \in \mathcal{F}} \dim \lambda'(U)$$

for all $U \subseteq I$.

If the family $\mathcal{F}$ is an irreducible algebraic variety, considered as a subset of L^I (A^I), then $LGP(\mathcal{F})$ $(AGP(\mathcal{F}))$ is open and everywhere dense in $\mathcal{F}$.

Definition. A polyhedral set $S \subset \mathbb{R}^n$ is called *simple* if either it is a simplex or $S = \bigcap_{i \in I_m} H^+(f_i)$, where $\{f_i\}_{i \in I_m} \in LGP$ and $m \le n$.

Let $a \neq b \in \mathbb{R}^n$, denote by $[ab]$, (ab), $\langle ab\rangle$, and $[ab\rangle$ the segment between a and b, its relative interior, the straight line passing through a and b, and the ray, emanating from a, passing through b. Also put

$$\rangle ab\rangle = [ab\rangle \setminus \{a\} \setminus (ab)\,.$$

This is the ray in $\langle ab\rangle$, emanating from b and not containing a.

Definition. A set $U \supseteq V$ is called V-*starshaped* if for any $u \in U$ and $v \in V$ we have $[uv] \subseteq U$. Clearly, if

(1) $V \neq \emptyset$, then a V-starshaped set U is starshaped;
(2) sets $U_i,\ i \in I$, are V-starshaped, then $\bigcup_{i \in I} U_i$ and $\bigcap_{i \in I} U_i$ are V-starshaped;
(3) U is V-starshaped and $W \subseteq V$, then U is W-starshaped.

Lemma 1. *Let X be a polyhedral set in an affine space L, and let $\{F_i\}_{i \in I_m}$ be its facets. Then there exists an affine embedding of L into an affine space L' of larger dimension and a simple polyhedral set S in L' such that $X = S \cap L$, $\operatorname{int} X = (\operatorname{int} S) \cap L$, and for any $i \in I_m$ $F_i = G_i \bigcap L$, where $\{G_i\}_{i \in I_m}$ are the facets of S. Also, if X is a polytope, then S is a simplex.*

Lemma 2. *Let G be a finite graph on vertices $V \subset L_1^n$ in general position, let $E = \{e_i\}_{i \in I_l}$, $l \ge |V|$ be its edges, and let $\mathcal{F}$ be the family of all maps $g : I_l \to L_1^n$ such that if $e \in E$ is the edge with endpoints f_1 and f_2, then $H(f_1) \cap H(f_2) \subseteq H(g(e))$. Then there exists a subset $W \subseteq V$ $(W \neq \emptyset)$ such that for any map $g \in LGP(\mathcal{F})$*

$$\bigcap_{i=1}^{l} H(g(e_i)) \subseteq \bigcap_{f \in W} H(f).$$

Lemma 3. *Let $\{V_i\}_{i=1}^d$ be a family of closed subsets of a simplex S in $\mathbb{R}^{d-1}$ with facets $\{F_i\}_{i=1}^d$ such that:*

(1) *V_i is F_i-starshaped for all $i \in I_d$;*
(2) *$S \subseteq \bigcup_{i=1}^d V_i$.*

Then $\bigcap_{i=1}^d V_i \neq \emptyset$.

Lemma 4. *Let $\{U_i\}_{i=1}^d$ be a family of open subsets of simplex S in $\mathbb{R}^{d-1}$ with facets $\{F_i\}_{i=1}^d$ such that:*

(1) *$U_i \bigcup \operatorname{rint} F_i$ is $\operatorname{rint} F_i$-starshaped for all $i \in I_d$;*
(2) *$\operatorname{int} S \subseteq \bigcup_{i=1}^d U_i$;*
(3) *any point $p \in \operatorname{bd} S$ has a neighborhood $N(p)$ such that $N(p) \cap \operatorname{int} S \subseteq \bigcup_{i,\, p \in F_i} U_i$.*

Then $\bigcap_{i=1}^d U_i \neq \emptyset$.

3. Reduction of Theorems 1–3 to special cases

We now show that it is sufficient to prove Theorems 1–3 in some special cases.

Lemma 5. *It is sufficient to prove Theorems 1–3 for the case of a simple polyhedral set (polytope) X.*

Proof. By Lemma 1, $X = S \cap L$, where S is simple polyhedral set, L is an n-dimensional affine subspace in $\mathbb{R}^N$ and $F_i = F_i^S \bigcap L$, where $\{F_i^S\}_{i \in I_m}$ are the facets of S.

Indeed, Theorems 1–3 for S imply those for X, if we note that

$$\operatorname{conv}(\{a\} \cup F_i) = L \bigcap \operatorname{conv}(\{a\} \cup F_i^S)$$
$$\operatorname{int}\operatorname{conv}(A_i \cup F_i) = L \bigcap \operatorname{int}\operatorname{conv}(A_i \cup F_i^S)$$
$$\operatorname{int} X = L \bigcap \operatorname{int} S.$$

□

We now consider a simple polyhedral set S, let its facets be $\{F_i\}_{i \in I_m}$.

Denote $\mathcal{A}_1 = \mathbb{R}^{Nl}$ the affine space of all possible sets A in Theorem 1. In Theorems 2 and 3 the spaces of all possible sets A will be some $\mathcal{A}_2, \mathcal{A}_3 \subset \mathcal{A}_1$.

Lemma 6. *It is sufficient to prove Theorems 1–3 for any everywhere dense subset $\mathcal{B} \subseteq \mathcal{A}_i$ $(i = 1, 2, 3)$.*

Proof. In fact, let $A \in \mathcal{A}_i$, let the elements of A be numbered a_1 to a_l.

Take one of Theorems 1–3.

Suppose that the sequence $\{A^k\} \subseteq \mathcal{B}$ is such that

$$A^k = \{a_i^k\}_{i \in I_l} \quad \forall i \ \lim_k a_i^k = a_i.$$

Since $\{A^k\} \subseteq \mathcal{B}$, the theorem holds for A^k. Thus for any k we have a partition of A^k into A_i^k.

Obviously, there exists an infinite set N of positive integers such that the partitions of the indices I_l corresponding to the partitions of A^k $(k \in N)$ coincide.

Then

$$\operatorname{conv}(A_i \cup F_i) = \lim_k \operatorname{conv}(A_i^k \cup F_i)$$

and

$$\bigcap_{a\in A_i} \operatorname{conv}(a\cup F_i) = \lim_k \bigcap_{a\in A_i^k} \operatorname{conv}(a\cup F_i).$$

Since the sets $\operatorname{conv}(a\cup F_i)$ are closed, then if

$$x\notin \bigcup_{i\in I_m} \cap_{a\in A_i} \operatorname{conv}(a\cup F_i),$$

then if k is large enough

$$x\notin \bigcup_{i\in I_m} \cap_{a\in A_i^k} \operatorname{conv}(a\cup F_i).$$

So the case of Theorem 1 is considered.

In case of Theorems 2 and 3, if

$$x\in \operatorname{int}\operatorname{conv}(A_i\cup F_i)\bigcap \operatorname{int}\operatorname{conv}(A_j\cup F_j) \qquad i\neq j,$$

then if k is large enough

$$x\in \operatorname{int}\operatorname{conv}(A_i^k\cup F_i)\bigcap \operatorname{int}\operatorname{conv}(A_j^k\cup F_j) \qquad i\neq j.$$

In case of Theorem 3, if for some i

$$\operatorname{int}\operatorname{conv}(A_i\cup F_i)\bigcap \operatorname{int} S\neq\emptyset,$$

then for large enough k

$$\operatorname{int}\operatorname{conv}(A_i^k\cup F_i)\bigcap \operatorname{int} S\neq\emptyset.$$

□

Lemma 7. *It is sufficient to prove Theorem 1 for the case $A\subset \operatorname{int} S$.*

Proof. Remind that in this theorem S is a simplex. We construct a map $\pi(x) : \mathbb{R}^n\to S$ such that for every $x\in\mathbb{R}^n$ and any facet F_i of S the condition holds:

$$\operatorname{conv}(\{x\}\cup F_i)\supseteq \operatorname{conv}(\{\pi(x)\}\cup F_i).$$

For any $x\in S$ put $\pi(x)=x$. Otherwise, put

$$I=\{i\in I_m : f_i(x)\geq 0\} \quad I' = I_m\setminus I,$$

in other words, $\{\operatorname{aff} F_i\}_{i\in I'}$ are the hyperplanes that separate x from S. Both sets I and I' are not empty. Denote $M=\bigcap_{i\in I} F_i$ and $N=\bigcap_{i\in I'} F_i$.

It is clear that $x\notin \operatorname{aff} M$ and $x\notin\operatorname{aff} N$. Moreover, it can be easily checked that there exists only one pair $\rho(x)\in\operatorname{aff} M, \pi(x)\in\operatorname{aff} N$ such that the points $\{\rho(x),\pi(x),x\}$ lie on the same line in that order.

Note that for every $i\in I$ $\rho(x)\in F_i$ and for every $i\in I'$ $\pi(x)\in F_i$, hence for every $i\in I_m$ we have $\operatorname{conv}(\{x\}\cup F_i)\supseteq\operatorname{conv}(\{\pi(x)\}\cup F_i)$

Now Theorem 1 for A and S follows from Theorem 1 for $\pi(A)$ and S.

By the Lemma 6, if we have to consider the case $A\subset S$, it is sufficient to prove Theorem 1 for the case $A\subset\operatorname{int} S$. □

By Lemma 6 we may also consider sets A such that $A\in AGP$ and $a\notin\operatorname{aff} F_i$ for all $a\in A$, $j\in I_m$. The family of sets A such that $A\in AGP$ and $A\cap\operatorname{aff} F_i=\emptyset$ for all i is open and everywhere dense in $A\in\mathcal{A}_1$.

Besides, we may impose another condition. We need some notation to formulate it.

Let S be a simple polyhedral set, let $\{F_i\}_{i\in I_m}$ be its facets, let A be $A = \{a_i\}_{i\in I_l}$. Put by definition

$$\mathcal{I} = \{(i,j,k) \in I_l \times I_m \times I_m : j > k\},$$

denote by $\mathcal{F}(S,A)$ the family of all maps $g : \mathcal{I} \to L_1^N$ such that

$$H(g(i,j,k)) = H(g_{ijk}) \supseteq \operatorname{aff}(a_i, F_j \cap F_k),$$

denote by $\mathcal{G}(S)$ the family of all maps $\gamma : \mathcal{I} \to L_1^N$ such that

$$H(\gamma_{ijk}) \supseteq F_j \cap F_k \qquad \bigcap_{j>k} H(\gamma_{ijk}) \neq \emptyset.$$

Clearly, $\mathcal{F}(S,A) \subseteq \mathcal{G}(S)$ and for any $g \in \mathcal{G}(S)$ there exists A such that $g \in \mathcal{F}(S,A)$.

Thus the family of all sets A such that $\mathcal{F}(S,A) \cap LGP(\mathcal{G}(S)) \neq \emptyset$ is everywhere dense in the variety of all sets A.

Thus we will prove Theorems 1–3 with the following assumptions:

(1) S is a simple polyhedral set with facets $\{F_i\}_{i\in I_m}$;

(2) $A \in AGP$ and $a \notin \operatorname{aff} F_i$ for all $a \in A$, $i \in I_m$;

(3) $\mathcal{F}(S,A)$ contains some $g \in LGP(\mathcal{G}(S))$;

(4) in Theorem 1 $A \subset \operatorname{int} S$.

For a given $g \in LGP(\mathcal{G}(S))$ we will write g_{ajk} instead of g_{ijk}, if the number of elements $a \in A$ equals i. Note that under the above conditions

$$H(g_{ajk}) = \operatorname{aff}(a, F_j \cap F_k),$$

since

$$\operatorname{aff}(a, F_j \cap F_k) \subseteq H(g_{ajk}) \quad \text{and} \quad \dim_a H(g_{ajk}) = \dim_a \operatorname{aff}(a, F_j \cap F_k).$$

4. Proof of Theorems 1–3

Now we are ready to prove Theorems 1–3.

Proof of Theorem 1. In this theorem S is a simplex. It was noted in the previous section that we only have to consider the case $A \subset \operatorname{int} S$.

We use induction on $\dim_a S$. In the case $\dim_a S = 1$ the theorem is obvious.

Denote for all $a \in A$ and $i \in I_m$

$$V_{ai} = \operatorname{conv}(\{a\} \cup \operatorname{rint} F_i) \setminus \{a\}$$

and for all $i \in I_m$

$$U_i = \bigcup_{A'\subseteq A,\ |A'|>=l_i} \cap_{a\in A'} V_{ai}.$$

It is obvious that the sets U_i are open and $x \in \operatorname{int} S \cap U_i$ iff there are at least l_i points $a \in A$ such that $x \in V_{ai}$, in other words, for at least l_i points $a \in A$ there exists $y \in \operatorname{rint} F_i$ such that $x \in (ay)$. We show that $\{U_i\}_{i\in I_m}$ satisfy the conditions of Lemma 4.

Condition (1) holds. Indeed, let $x \in U_i$ and for some $y \in \operatorname{rint} F_i$ $x' \in (xy)$. Let $x \in V_{ai}$, then there exists $z \in \operatorname{rint} F_i$ such that $x \in (az)$. Then $\langle ax'\rangle \cap [zy] = z' \in \operatorname{rint} F_i$ and $x' \in (az')$.

Thus $x' \in V_{ai}$. Considering all $a \in A'$ we obtain $x' \in U_i$.

We now show that condition (3) holds. Let $p \in \operatorname{bd} S$, $I = \{i : p \in F_i\}$ and $I' = I_m \setminus I$. Condition (3) holds iff for any $x \in \operatorname{int} S \cap N(p)$ there exists $i \in I$ such that for at least l_i points $a \in A$ we have $\rangle ax\rangle \cap \operatorname{rint} F_i \neq \emptyset$.

For any $a \in A$ the point $y_a(x) = \rangle ax \rangle \cap \operatorname{bd} S$ is a continuous function of $x \in S$. Since $y_a(p) = p,\ a \in A$, there exists a neighborhood $N(p)$ of p such that for any $x \in N(p)$, $a \in A$, and $i \notin I$ we have $y_a(x) \notin F_i$.

We show that

$$N(p) \bigcap \operatorname{int} S \subseteq \cup_{i\in I} U_i.$$

To prove it we should show that if $x \in N(p) \cap \operatorname{int} S$, then for some $i \in I$ we have at least l_i points $a \in A$ such that $y_a(x) \in \operatorname{rint} F_i$. In this case for such a we have $x \in (ay_a(x))$ and therefore $x \in U_i$.

Assume the contrary that for all $i \in I$

$$|\{y_a(x)\}_{a\in A} \bigcap \operatorname{rint} F_i| < l_i,$$

or, equivalently,

$$|\{y_a(x)\}_{a\in A} \bigcap \operatorname{rint} F_i| \le l_i - 1.$$

Then for at least $|I|$ points $a \in A$ (denote them by A') we have

$$y_a(x) \in F_{s(a)} \bigcap F_{t(a)}, \quad s(a), t(a) \in I.$$

Hence

$$x \in \bigcap_{a\in A'} \operatorname{aff}(a, F_{s(a)} \cap F_{t(a)}),$$

and therefore

$$\operatorname{aff}(a, F_{s(a)} \cap F_{t(a)}) = H(g_{as(a)t(a)}),$$

where g_{ast} is defined above, and

$$\{\operatorname{aff} F_{s(a)}, \operatorname{aff} F_{t(a)}\}_{a\in A'} \subseteq \{H(f_i)\}_{i\in I}.$$

Denote the family of maps

$$\mathcal{G}' = \{\gamma : A' \to L_1^N : H(\gamma(a)) \supseteq F_{s(a)} \cap F_{t(a)}\}.$$

Since the map of restriction $\mathcal{G}(S) \to \mathcal{G}'$ is surjective, the restriction $g' : a \mapsto g_{as(a)t(a)}$ of $g \in LGP(\mathcal{G}(S))$ must be in $LGP(\mathcal{G}')$.

We apply Lemma 2 to the graph with vertices $\{f_i\}_{i\in I}$, edges $(f_{s(a)}, f_{t(a)}),\ a \in A'$ and the map g'. The conditions of the lemma hold since $|A'| \ge |I|$. By Lemma 2 for some i we have

$$\bigcap_{a\in A'} H(g'_a) \subseteq H(f_i) = \operatorname{aff} F_i,$$

and

$$x \in \bigcap_{a\in A'} H(g'_a) \subseteq \operatorname{aff} F_i \qquad x \in \operatorname{int} S.$$

This is a contradiction, hence condition (3) holds.

Consider condition (2).

As above if for some $x \in \operatorname{int} S,\ y_a(x) \notin \operatorname{rint} F_i$ for all $i \in I_m, a \in A$, then

$$x \in \bigcap_{a\in A} \operatorname{aff}(a, \operatorname{aff} F_{s(a)} \cap \operatorname{aff} F_{t(a)}).$$

Applying Lemma 2 to the graph with vertices $\{f_i\}_{i\in I_m}$ and edges $(f_{s(a)}, f_{t(a)})$ $(a \in A)$ we obtain that $x \in \operatorname{aff} F_i \cap \operatorname{int} S$ for some i. This is a contradiction.

Finally we apply Lemma 4 and obtain $x \in \bigcap_{i\in I_m} U_i$. Then for any $i \in I_m$ there are at least l_i points $a \in A$ such that $x \in V_{ai}$.

For any $a \in A$ there exists at most one i such that $x \in V_{ai}$, then for any $i \in I_m$ there exists a set $A_i \subseteq A$ with l_i such that for all $a \in A_i$ $x \in V_{ai}$. The sets A_i are obviously disjoint.

Thus we have

$$x \in \operatorname{conv}(\{a\} \cup \operatorname{rint} F_i) \quad \text{for all} \quad i \in I_m, a \in A_i.$$

For all $i \in I_m$ and $a \in A_i$

$$S \subseteq \bigcup_{i \in I_m} \operatorname{conv}(\{x\} \cup F_i) \quad \text{and} \quad \operatorname{conv}(\{a\} \cup F_i) \supseteq \operatorname{conv}(\{x\} \cup F_i),$$

then $S \subseteq \bigcup_{i \in I_m} \cap_{a \in A_i} \operatorname{conv}(\{a\} \cup F_i)$ and the theorem is proved. □

Proof of Theorem 2. First consider the case when S is a simplex. Denote

$$V_i = \{x \in S : |\operatorname{int} \operatorname{conv}(\{x\} \cup F_i) \bigcap A| < l_i\}.$$

Obviously, the sets $\{V_i\}_{i \in I_m}$ are closed. Also, if $x \in V_i$, $y \in F_i$, and $x' \in [xy]$, then $\operatorname{int} \operatorname{conv}(\{x'\} \cup F_i) \subseteq \operatorname{int} \operatorname{conv}(\{x\} \cup F_i)$. Hence V_i are F_i-starshaped.

Consider two cases:

Case 1: $S \subseteq \bigcup_{i \in I_m} V_i$. By Lemma 3 there exists $x \in \bigcap_{i \in I_m} V_i$. It means that there are at least m points $a \in A$ such that $a \notin \operatorname{int} \operatorname{conv}(\{x\} \cup F_i)$ for all i, denote the set of such points a by A'. Then for all $a \in A'$

$$a \in \operatorname{conv}(\{x\} \cup (\operatorname{aff} F_{s(a)} \cap \operatorname{aff} F_{t(a)})).$$

Let $I = \{i : x \notin F_i\}$, then $s(a), t(a) \in I$ for all $a \in A'$, otherwise we would have $a \in F_i$, which is not true by assumption (2). In other words,

$$x \in \bigcap_{a \in A'} \operatorname{aff}(a, \operatorname{aff} F_{s(a)} \cap \operatorname{aff} F_{t(a)}).$$

As in the proof of Theorem 1 we apply Lemma 2 taking $\{f_i\}_{i \in I}$ as vertices, $(f_{s(a)}, f_{t(a)})$, $a \in A'$ as edges, and the restriction $g'(a) = g_{as(a)t(a)}$ of the map g. By this lemma we have $x \in \operatorname{aff} F_i$, $i \in I$, this is a contradiction with the definition of I.

Thus case 1 is impossible.

Case 2: $S \not\subseteq \bigcup_{i \in I_m} V_i$. Let $x \in S \setminus \bigcup_{i \in I_m} V_i$. Then for any $i \in I_m$ there exist at least l_i points $a \in A$ such that $a \in \operatorname{int} \operatorname{conv}(\{x\} \cup F_i)$.

The sets $\operatorname{int} \operatorname{conv}(\{x\} \cup F_i)$ $(i \in I_m)$ do not intersect, hence each of them contains exactly l_i points of A, so we denote

$$A_i = A \bigcap \operatorname{int} \operatorname{conv}(\{x\} \cup F_i).$$

Then for any $i \in I_m$

$$\operatorname{conv}(A_i \cup F_i) \subseteq \operatorname{conv}(\{x\} \cup F_i),$$

hence for all $i \neq j \in I_m$

$$\operatorname{int} \operatorname{conv}(A_i \cup F_i) \bigcap \operatorname{int} \operatorname{conv}(A_j \cup F_j) = \emptyset.$$

Thus the proof is complete if S is a simplex.

Now suppose S is not a simplex, then $\bigcap_{i \in I_m} F_i \neq \emptyset$. Consider two cases.

Case 1: $\bigcap_{i \in I_m} F_i = \{v\}$. Obviously, there exists a hyperplane H such that $S' = S \cap H$ is a simplex with facets $F'_i = F_i \cap H$, $i \in I_m$.

Let the central projection from v to H take $a \in A$ to $p(a)$. The theorem is already proved for $A' = p(A)$ and simplex S', thus there is a partition $A = \cup_{i\in I_m} A_i$ such that $|A_i| = l_i$ and for all $i \neq j \in I_m$

$$\operatorname{int}\operatorname{conv}(p(A_i) \cup F'_i) \bigcap \operatorname{int}\operatorname{conv}(p(A_j) \cup F'_j) = \emptyset.$$

Take the cone $C_i = \operatorname{int}\operatorname{conv}(F_i \bigcup \cup_{a\in A_i} [va\rangle)$ formed by the rays from v that intersect $\operatorname{int}\operatorname{conv}(p(A_i) \cup F'_i)$ without the point v. Obviously, we have for all $i \neq j \in I_m$

$$C_i \cap C_j = \emptyset$$

and for all $i \in I_m$

$$\operatorname{int}\operatorname{conv}(A_i \cup F_i) \subseteq \operatorname{int}\operatorname{conv}(F_i \bigcup \cup_{a\in A_i} [va\rangle) = C_i.$$

Then for all $i \neq j \in I_m$

$$\operatorname{int}\operatorname{conv}(A_i \cup F_i) \bigcap \operatorname{int}\operatorname{conv}(A_j \cup F_j) = \emptyset.$$

In this case the proof is complete.

Case 2: $\dim_a L > 0$, where $L = \cap_i F_i$. Let p be a projection along L. Then $p(L)$ is a point. The previous case gives a partition $A = \cup_{i\in I_m} A_i$ such that for all $i \neq j \in I_m$

$$\operatorname{int}\operatorname{conv}(p(A_i) \cup p(F_i)) \bigcap \operatorname{int}\operatorname{conv}(p(A_j) \cup p(F_j)) = \emptyset.$$

Then it is clear that

$$\operatorname{int}\operatorname{conv}(A_i \cup F_i) \bigcap \operatorname{int}\operatorname{conv}(A_j \cup F_j) = \emptyset$$

and the theorem is proved. □

Proof of Theorem 3. For any $i \in I_m$ put

$$U_i = \{x \in \operatorname{int} S : [xa] \cap \operatorname{rint} F_i \neq \emptyset \quad \text{for at least } l_i \text{ points } a \in A\}.$$

The sets U_i are open.

We show that $\{U_i\}_{i\in I_m}$ satisfy the conditions of Lemma 4.

Condition (1) holds. Let for some $a \in A$ $x \in U_i$, $y \in \operatorname{rint} F_i$, $x' \in [xy]$, and $z = [xa] \cap \operatorname{rint} F_i$. The points x' and a are in different halfspaces w. r. t. the hyperplane $\operatorname{aff} F_i$, hence there exists $z' = [x'a] \cap \operatorname{aff} F_i$, $z' \in [zy]$, and $z' \in \operatorname{rint} F_i$.

Consider condition (3). Let

$$p \in \operatorname{bd} S, \quad I = \{i : p \in F_i\}.$$

We prove that there is a neighborhood $N(p)$ of p such that

$$N(p) \bigcap \operatorname{int} S \subseteq \bigcup_{i\in I} U_i,$$

or, equivalently, for any $x \in \operatorname{int} S \cap N(p)$ for some $i \in I$ at least l_i of the segments $[x_j x]$ intersect $\operatorname{rint} F_i$.

Let $A' = \{a \in A : a \in \bigcup_{i\in I} H_i\}$, in this theorem $|A| \geq |I|$ and $[ap] \cap \operatorname{bd} S = \{p\}$ for all $a \in A'$. Take any $a \in A'$ and $x \in S$ and let $y_a(x)$ be the farthest from x point in $[ax] \cap \operatorname{bd} S$.

Obviously, $y_a(x)$ is a continuous function of $x \in S$. Since $y_a(p) = p$ for all $a \in A'$, then there exists a neighborhood $N(p)$ of p such that for all $x \in N(p)$, $a \in A'$, and $i \notin I$ we have $y_a(x) \notin F_i$.

We shall show that for any $x \in N(p) \cap \operatorname{int} S$ there exists $i \in I$ such that $x \in U_i$. Equivalently, at least l_i of the points $y_a(x)$ $(a \in A')$ are in $\operatorname{rint} F_i$ for some $i \in I$.

Assume the contrary. Then we have at least $|I|$ points $a \in A'$ such that $y_a(x) \in F_{s(a)} \cap F_{t(a)}$ for some $x \in N(p) \cap \operatorname{int} S$. Denote the set of these a by A''. Then

$$x \in \bigcap_{a \in A''} \operatorname{aff}(a, \operatorname{aff} F_{s(a)} \cap \operatorname{aff} F_{t(a)}),\ s(a), t(a) \in I.$$

As in the proof of Theorem 1 by Lemma 2 we get a contradiction, since $|A''| \geq |I|$.

We now show that condition (2) holds. We show that $\operatorname{int} S \subseteq \bigcup_{i \in I_m} U_i$. Assume the contrary: there exists x such that for any $i \in I_m$ at least l_i of the points $y_a(x)$ are in $\operatorname{rint} F_i$. Denote the set of these points a by A''. As above we have $x \in \bigcap_{a \in A''} \operatorname{aff}(a, \operatorname{aff} F_{s(a)} \cap \operatorname{aff} F_{t(a)})$. That is a contradiction, since $|A''| \geq m + 1$.

If S is a simplex, then $\bigcap_{i \in I_m} U_i \neq \emptyset$ by Lemma 4.

If $\bigcap_{i \in I_m} F_i = \{v\}$, then as above we take a hyperplane H such that $S' = S \cap H$ is a simplex with facets $F'_i = F_i \cap H,\ i \in I_m$. Applying Lemma 4 to S' and the sets $U'_i = U_i \cap H$ we obtain

$$\bigcap_{i \in I_m} U'_i \neq \emptyset \quad \text{and} \quad \bigcap_{i \in I_m} U_i \neq \emptyset.$$

Take some $x \in \bigcap_{i \in I_m} U_i$ and put

$$A_i = \{a \in A : y_a(x) \in \operatorname{rint} F_i\}.$$

Then by the definition of U_i we have $|A_i| \geq l_i$. Also, $\sum_{i \in I_m} l_i = |A|$ and the sets A_i do not intersect pairwise, then $|A_i| = l_i$ for all $i \in I_m$ and the sets A_i form some partition of A.

Let R_i be a cone of rays $[xy\rangle$, where $y \in \operatorname{rint} F_i$ without the point x. $R_k \cap R_l = \emptyset$ for all $k \neq l \in I_m$ and since for all $a \in A_i$ we have $\operatorname{rint} F_i \ni y_a(x) \in [ax]$, then $A_i \subset R_i$. Hence $\operatorname{int} \operatorname{conv}(A_i \cup F_i) \subseteq R_i$ and

$$\operatorname{int} \operatorname{conv}(A_k \cup F_k) \bigcap \operatorname{int} \operatorname{conv}(A_l \cup F_l) = \emptyset,\ k \neq l \in I_m.$$

Since A_i and S are on different halfspaces w. r. t. $\operatorname{aff} F_i$, then for any $i \in I_m$

$$\operatorname{int} \operatorname{conv}(A_i \cup F_i) \bigcap \operatorname{int} S = \emptyset.$$

The last case is when $\dim_a \bigcap_{i \in I_m} F_i > 0$, let $L = \bigcap_{i \in I_m} F_i$.

As in the previous proof take a projection p along L. By the same argument we can deduce the theorem for this case from this theorem for the projection $p(S)$, which is already proved. □

5. Proof of Theorems 4–6 and Corollary 1

As above we apply Lemma 1 to X. Thus we have a simplex S in $\mathbb{R}^{m-1}$ and an n-dimensional affine subspace $L \subseteq \mathbb{R}^{m-1}$ such that $X = L \cap S$, $F_i = L \cap G_i$, $\operatorname{int} X = L \cap \operatorname{int} S$, and $\operatorname{int} F_i = L \cap \operatorname{int} G_i$.

For any point $s \in S$ and $i \in I_m$ put $B_i(s) = \operatorname{conv}(G_i \cup \{s\})$. Note that for any $s \in \operatorname{int} S$ the sets $B_i(s)$ give a partition of S. To prove Theorems 4 and 6 we consider partitions of X into sets $A_i = B_i(s) \cap L$ for some $s \in \operatorname{int} S$. Obviously, $A_i(s) \cap \operatorname{bd} X = F_i$ for any $s \in \operatorname{int} S$.

We need a lemma:

Lemma 8. *Under the above notation $A_i(s)$ are continuous functions of $s \in S$ in the Hausdorff metric, and for any $i \in I_m$ and $s \in S$, $s \notin G_i$ we have* $\operatorname{int} B_i(s) \cap L \neq \emptyset$.

Proof. Consider the second statement of the lemma. Assume the contrary:

$$\operatorname{int} B_i(s) \cap L = \emptyset,$$

then $L \cap \operatorname{int} G_i \neq \emptyset$ implies that $L \subseteq \operatorname{aff} G_i$. But $L \cap G_j \neq \emptyset$ for some $j \neq i$ and $\operatorname{int} G_j \cap \operatorname{aff} G_i = \emptyset$, that is a contradiction.

Since $\operatorname{int} B_i(s) \cap L \neq \emptyset$ and $B_i(s)$ is a continuous function of s, the first statement of the lemma is true for points $s \notin G_i$.

For points $s \in G_i$ we may prove the continuity by definition: let $s_l \to s$, then, obviously,

$$\forall l \ A_i(s_l) \supseteq F_i, \quad \lim_l A_i(s_l) = F_i = A_i(s).$$

□

Now we are ready to prove the theorems.

Proof of Theorems 4 and 5. Put

$$U_i = \{s \in S : \mu(A_i(s)) \leq \mu_i\}.$$

We show that Lemma 3 can be applied to S and sets U_i.

The functions $A_i(s)$ and $\mu(A)$ are continuous, then the sets U_i are closed.

If $f \in F_i$, $s' \in [fs]$, then $A_i(s') \subseteq A_i(s)$, and therefore $s' \in U_i$. It means that the set U_i is F_i-starshaped.

The sets U_i cover S; otherwise we could find a point $s \in S$ such that for any $i \in I_m$ $\mu(A_i(s)) > \mu_i$, the sets $A_i(s)$ form a partition of X, so summing up the inequalities we obtain $\mu(X) > 1$, this is a contradiction.

Thus Lemma 3 gives a point $s^* \in \cap_{i \in I_m} U_i$. For the partition we have:

$$\mu(A_i(s^*) \leq \mu_i,$$

after summation over $i \in I_m$ we have 1 on both sides, hence in fact

$$\mu(A_i(s^*)) = \mu_i.$$

Now the proof becomes different for Theorems 4 and 5.

Case of Theorem 5: Obviously, $A_i(s^*) \supseteq F_i$ for all $i \in I_m$ and if $i \notin I$ ($\mu_i = 0$), then $A_i(s^*) = F_i$. Hence X is covered by $\{A_i(s^*)\}_{i \in I}$.

Case of Theorem 4: $\mu(A_i(s^*)) > 0$ for all $i \in I_m$, then $s^* \in \operatorname{int} S$ and $A_i(s^*) \cap \operatorname{bd} X = F_i$ for all $i \in I_m$.

□

Proof of Theorem 6. Put

$$U_i = \{s \in S : \operatorname{int} A_i(s)) \cap V = \emptyset\}.$$

We show that Lemma 3 can be applied to S and the sets U_i.

If $s \notin U_i$, then for some $v \in V$ we have $\operatorname{int} A_i(s) \ni v$, for any s' in some neighborhood of s we still have $\operatorname{int} A_i(s) \ni v$. Thus the complement of U_i is open and U_i is closed.

If $f \in F_i$, $s' \in [fs]$, then $A_i(s') \subseteq A_i(s)$, and hence $s' \in U_i$. Thus the set U_i is F_i-starshaped, The sets U_i cover S; otherwise we would have a point $s \in S$ such that for any $i \in I_m$ $|\operatorname{int} A_i(s) \cap V| \geq 1$. The sets $A_i(s)$ give a partition of X, then $|V| \geq m$, which is a contradiction.

Thus Lemma 3 gives $s^* \in \cap_{i\in I_m} U_i$.

We show that we may assume $s^* \in \operatorname{int} S$. Note that every set U_i contains a neighborhood of G_i. Now let M be a face of S with maximum $\dim M$ and $M \ni s^*$, let M' be the face spanned by the vertices of S not contained in M. Let $m' \in \operatorname{rint} M'$, then the points $s' \in (s^* m')$ close enough to s^* will be contained in those of the sets U_i that contain M (and therefore a neighborhood of M). Those sets U_i that do not contain M, contain M', in this case $m' \in G_i$ and $s' \in U_i$ due to starshapedness. Thus we have $s' \in \operatorname{int} S \bigcap \cap_{i\in I_m} U_i$.

For the partition we have what we need

$$\forall i \in I_m \quad \operatorname{int} A_i(s^*) \cap V = \emptyset.$$

□

We need another lemma to prove Corollary 1.

Lemma 9. *Let $W \subset \mathbb{R}^n$ be a finite set in general position, $|W| \geq n+1$, let F be a facet of* $\operatorname{conv} W$*. Then there exists a possibly non-convex polytope with vertices W, having F as a facet.*

Proof. We use induction over $|W|$. The case $|W| = n+1$ is obvious.

If $\operatorname{int}\operatorname{conv} W \cap W = \emptyset$, then $\operatorname{conv} W$ is the polytope we need.

Otherwise, let F' be another facet of $\operatorname{conv} W$ and

$$W' = (F' \cup \operatorname{int}\operatorname{conv} W) \cap W.$$

Clearly, $W' \subseteq W$ and W' does not contain one of the vertices of F, otherwise F and F' would coincide. Besides, $F' \cap W$ contains at least n points, and $\operatorname{int}\operatorname{conv} W \cap W$ contains at least one point, hence $|W'| \geq n+1$.

Let us apply the inductive assumption to W' and F'. We obtain a polytope Y' with vertices W'. Put $Y = \operatorname{conv} W \setminus (\operatorname{int} Y' \cup \operatorname{rint} F')$, this is a polytope, with vertices W, having F as a facet. □

Proof of Corollary 1. Put $X = \operatorname{conv} W$. Let $\{F_i\}_{i\in I_m}$ be the set of facets of X. Applying Theorem 6 to X and the set V, we obtain a partition of X into sets A_i such that $A_i \cap \operatorname{bd} X = F_i$ and $V \cap \operatorname{int} A_i = \emptyset$ for all $i \in I_m$.

Let us partition $W \cap \operatorname{int} X$ into sets W_i so that $W_i \subset A_i$ for all i. Denote $W_i' = W \cap F_i$.

Consider indices $i \in I_m$ such that $W_i \neq \emptyset$. Then we apply Lemma 9 to the set $W_i \cup W_i'$ and the facet F_i, so we have a polytope Y_i with vertices W_i. Note that $Y_i \subseteq A_i$, therefore $\operatorname{int} Y_i \cap V = \emptyset$, and $Y_i \cap V = \emptyset$ due to general position.

Put $Y = X \setminus (\bigcup_{i\in I_m, W_i\neq\emptyset} (\operatorname{int} Y_i \cup \operatorname{rint} F_i))$. Then the set of vertices of Y is W, and $Y \supset V$, since for all $i \in I_m$, $W_i \neq \emptyset$ we have $Y_i \cap V = \emptyset$.

□

6. Corollary for partitions of $\mathbb{R}^n$

We should define some properties for a family of closed convex sets $\{V_i\}_{i\in I_m}$, $V_i \subseteq L$, where L is a linear space. Some of these properties were already discussed in [1].

Property 1. For any set of m vectors $v_i \in L$ $(i \in I_m)$ there exists a permutation $\sigma \in S^m$ such that

$$\bigcup_{i\in I_m} (V_i + v_{\sigma(i)}) = L.$$

Property 2. For any set of m vectors $v_i \in L$ $(i \in I_m)$ there exists a permutation $\sigma \in S^m$ such that the sets

$$V_i' = V_i + v_{\sigma(i)}$$

do not intersect pairwise by interior points.

We also need two definitions:

Definition. A partition of $V \subseteq \mathbb{R}^n$ into V_i $(i \in I_m)$ is called an *affine partition*, if there are affine functions $\lambda_i \in L_1^n$ $(i \in I_m)$ such that

$$V_i = \{x \in V : \forall j \in I_m \; l_i(x) \geq l_j(x)\}.$$

Definition. A partition of $V \subseteq \mathbb{R}^n$ into V_i $(i \in I_m)$ is called a *hierarchically affine partition*, if this is either an affine partition, or the set I_m can be partitioned into subsets $J_1, J_2, \dots, J_k$ $(k \geq 2)$ such that the sets $V_j' = \cup_{i\in J_j} V_i$ $(j \in I_k)$ form an affine partition of V, and for all $j \in I_k$ the sets V_i $(i \in J_j)$ form a hierarchically affine partition of V_j'.

We define some stronger variants of Properties 1 and 2 that can be used in recursive proofs for hierarchically affine partitions:

Property 3. For any set of positive integers l_i $(i \in I_m, \sum_{i\in I_m} l_i = l)$ and any set of vectors $A \subset L$ $(|A| = l)$ there exists a partition of A into A_i $(i \in I_m)$, satisfying the following conditions:

1) $|A_i| = l_i$ for all $i \in I_m$;

2) The sets $W_i = \cap_{a\in A_i}(V_i + a) = V_i \overset{*}{-} (-A_i)$ $(i \in I_m)$ cover L, i.e.

$$\bigcup_{i\in I_m} W_i = L.$$

Property 4. For any set of positive integers l_i $(i \in I_m, \sum_{i\in I_m} l_i = l)$ and any set of vectors $A \subset L$ $(|A| = l)$ there exists a partition of A into A_i $(i \in I_m)$, satisfying the following conditions:

1) $|A_i| = l_i$ for all $i \in I_m$;

2) The sets $W_i = \operatorname{conv} \cup_{a\in A_i}(V_i + a) = V_i + \operatorname{conv} A_i$ $(i \in I_m)$ do not intersect pairwise by interior points, i.e. for all $i \neq j \in I_m$

$$\operatorname{int} W_i \cap \operatorname{int} W_j = \emptyset.$$

Note that Properties 1 and 2 are contained in Properties 3 and 4 respectively, if we put $l_i = 1$ for all $i \in I_m$.

We formulate a corollary of Theorems 1 and 2:

Corollary 2. *Hierarchically affine partitions of* $\mathbb{R}^n$ *have Properties 3 and 4.*

We need some lemmas to prove this corollary:

Lemma 10. *If a family of closed convex sets $V_i \subseteq L$ may be factored modulo some linear subspace $M \subseteq L$, equivalently, $V_i = V_i + M$ $(i \in I_m)$, then the family of sets $\pi(V_i)$ has Property 1, 2, 3, or 4 in L/M, where $\pi : L \to L/M$ is a projection, iff the family $V_i \subseteq L$ has the same property.*

Lemma 11. *If a family of closed convex sets $V_i \subseteq L$ has one of Properties 1 and 3 in L, and for a given subspace $M \subseteq L$ and any $i \in I_m$ $V_i \cap M \neq \emptyset$, then the family of sets $V_i \cap M$ has the same property in M.*

Lemma 12. *If a family of closed convex sets $V_i \subseteq L$ has one of Properties 2 and 4 in L, and for a given subspace $M \subseteq L$ and any $i \in I_m$ we have $V_i \cap M \neq \emptyset$ and $\operatorname{int}(V_i \cap M) = (\operatorname{int} V_i) \cap M \neq \emptyset$, then the family of sets $V_i \cap M$ has the same property in M.*

The proof of these three lemmas is trivial.

Lemma 13. *Suppose $V_i^n \subseteq L$ ($i \in I_m$, $n \in \mathbb{Z}^+$) are closed convex sets. Let for each i the sequence $\{V_i^n\}_{n\in\mathbb{Z}^+}$ converge to a closed convex set $V_i \subseteq L$. If for all $n \in \mathbb{Z}^+$ the family $\{V_i^n\}_{i\in I_m}$ has one of Properties 1, 2, 3, and 4, then $\{V_i\}_{i\in I_m}$ has the same property.*

Remark. The limit in the above lemma is considered in the family of metrics

$$\operatorname*{dist}_R(A, B) = \operatorname*{dist}_H(A \cap B_0(R), B \cap B_0(R)),$$

where dist_H is the Hausdorff metric, and $B_0(R)$ is a ball with radius R and center in 0.

Proof. Consider some sets A and $\{l_i\}_{i\in I_m}$. In case of Properties 1 and 2 we simply put $l_1 = l_2 = \ldots = l_m = 1$, so these properties are considered too.

We show that $\{V_i\}_{i\in I_m}$ has Property 3 or 4.

Since each family $\{V_i^n\}_{i\in I_m}$ has the same property, we apply its definition to A and $\{l_i\}_{i\in I_m}$.

So for any n we have a partition of A, taking some subsequence of families we may assume that the partition of A is the same for all n.

Now the Property 4 holds because

$$V_i + \operatorname{conv} A_i = \lim_n (V_i^n + \operatorname{conv} A_i).$$

In the case of Property 3 we may assume the contrary: let there exist a point x such that for any $i \in I_m$ $x \notin V_i \stackrel{*}{-} (-A_i)$. Then since the sets V_i are closed, we may find $\varepsilon > 0$ such that $\operatorname{dist}(x, V_i + a_i) > \varepsilon$ for all $i \in I_m$ and some $a_i \in A_i$. It means that for large enough n we have $\operatorname{dist}(x, V_i + a_i) > \frac{\varepsilon}{2}$, and therefore $x \notin V_i^n \stackrel{*}{-} (-A_i)$ for any $i \in I_m$ and large enough n. This contradicts the choice of partition $\{A_i\}_{i\in I_m}$ in Property 3 for the family $\{V_i^n\}_{i\in I_m}$. ☐

Proof of Corollary 2. First we consider the following case: the partition of L into V_i ($i \in I_m$) is affine, $m = \dim L + 1$, and the system of equations

$$\lambda_1(x) = \lambda_2(x) = \ldots = \lambda_m(x)$$

has only one solution, without loss of generality the solution is the origin. In this case λ_i is a linear function on L for all $i \in I_m$.

For any $i \in I_m$ the solution of the system of equations

$$\forall k, l \neq i \quad \lambda_k(x) = \lambda_l(x)$$

is a straight line. The subset of this line given by the inequality $\lambda_i(x) < \lambda_k(x)$ for any $k \neq i$ (it is the same for any $k \neq i$) is not contained in V_i, so we choose some vector $s_i \neq 0$ from it.

Thus V_i is a simplicial cone spanned by the vectors $s_1, \dots, s_m$ except s_i, while vectors s_i form a simplex S with facets F_i and their respective opposite vertices $\{s_i\}_{i\in I_m}$. This simplex contains the origin.

Consider a sequence of positive real numbers $t_n \to \infty$ such that for all $n \in \mathbb{N}$ the homothetic image $t_n S \supseteq A$.

So we can apply Theorem 1 or Theorem 2 to the set A, given numbers $\{l_i\}_{i\in I_m}$, and simplex $t_n S$. Taking some subsequence of $\{t_n\}$ if needed, we may assume that the partition of A is the same for all n.

Note that for all $a \in A$, $i \in I_m$

$$V_i + a = \lim_n \operatorname{conv}(\{a\} \cup t_n F_i).$$

By going to the limit as in the proof of Lemma 13 we show that Properties 3 and 4 hold in this case.

Now we consider the case of an affine partition into sets V_i such that the homogeneous components of degree 1 of the functions $\lambda_k - \lambda_1$ $(k = 2, \dots, m)$ are linearly independent.

In this case $m \le \dim L + 1$ and the system of equations

$$\lambda_1(x) = \lambda_2(x) = \ldots = \lambda_m(x)$$

has an affine subspace M as a solution, applying some translation we may assume that M is a linear subspace. Then $\{V_i\}_{i\in I_m}$ may be factored modulo M, after the factorization we have $m = \dim L' + 1$ $(L' = L/M)$ and this case is considered. By Lemma 10 the proof of this case is complete.

Now we consider a more general case: let for all $i \in I_m$ $\operatorname{int} V_i \neq \emptyset$.

If $\lambda_k - \lambda_1$ $(k = 2, \dots, m)$ are linearly independent, we take some $L' = L \oplus \mathbb{R}^{m-1}$, let the coordinates in the right summand be $x'_2, \dots, x'_m$. Then put $\lambda'_1 = \lambda_1$, $\lambda'_2 = \lambda_2 + x'_2, \dots, \lambda'_m = \lambda_m + x'_m$. The homogeneous components of degree 1 of the functions $\lambda'_k - \lambda'_1$ $(k > 1)$ are linearly independent. For the affine partition $\{V'_i\}_{i\in I_m}$ given by the functions $\{\lambda'_i\}_{i\in I_m}$ we have $V_i = L \cap V'_i$ and $\operatorname{int} V_i = (\operatorname{int} V'_i) \cap L \neq \emptyset$. Applying Lemmas 11 and 12 we complete the proof in this case.

Now consider an affine partition such that $\operatorname{int} V_i = \emptyset$ for some indices $i \in I_m$. Let $\{\lambda_i\}_{i\in I_m}$ be its affine functions. We show that we can make all $\operatorname{int} V_i$ nonempty by small changes of λ_i.

Take some positive integer n. Consider sets V_i such that their $\dim V_i$ is minimal, let the set of their indices be I. If this dimension is less than $\dim L$, then we can add some number $\varepsilon_1 < \frac{1}{n}$ to the functions $\{\lambda_i\}_{i\in I}$ so that the sets $\{V_i\}_{i\in I}$ get some nonempty interior, while neither of the $\dim V_i$ for $i \notin I$ decrease. The number ε_1 can be chosen small enough so that the new sets $\{V_i\}_{i\in I_m}$ will be in less than $\frac{1}{n}$ from their respective old sets in the Hausdorff metric.

Let us show that the number ε_1 can be chosen in more detail. If we choose some points $v_j \in \operatorname{rint} V_j$ for all $j \notin I$, we see that $\lambda_i(v_j) < \lambda_j(v_j)$ for all $i \in I, j \notin I$. Hence if ε_1 is small enough λ_i $(i \in I)$ the new sets V_j still contain v_j. If we consider some more points $v_j \in V_j$, we also show that for small enough ε_1 $\dim V_j$ $(j \notin I)$ do not decrease. The fact that the new sets $\{V_i\}_{i\in I}$ have nonempty interior and for small enough ε_1 the Hausdorff distance between new sets $\{V_i\}_{i\in I_m}$ and the respective old sets is less than $\frac{1}{n}$ is obvious.

Applying the above process for no more than $\dim L$ times we may add to some of λ_i some numbers $\varepsilon_k < \frac{1}{n}$ and finally have $\dim V_j = \dim L$ therefore $\operatorname{int} V_j \neq \emptyset$ for all $j \in I_m$. Denote the final partition by $\{V_i^n\}_{i\in I_m}$.

Each of the partitions $\{V_i^n\}_{i\in I_m}$ for different n has Properties 3 and 4, so by Lemma 13 the partition $\{V_i\}_{i\in I_m}$ has these properties too.

We have considered all cases for affine partitions.

Now we use induction over the number of the sets in a partition to prove the corollary for any hierarchically affine partition.

Consider a set of vectors A and a set of numbers $\{l_i\}_{i\in I_m}$. Let I_m be partitioned into J_j $(j \in I_k)$ and let $V'_j = \cup_{i\in J_j} V_i$.

We apply the statement of this corollary to the affine partition $\{V'_j\}_{j\in I_k}$, the set A, and the numbers $l'_j = \sum_{i\in J_j} l_i$. So the set A can be partitioned into A'_j $(j \in I_k)$ such that in case of Property 3

$$\bigcup_{j\in I_k} \cap_{a\in A'_j}(V'_j + a) = L,$$

or in case of Property 4

$$\mathrm{int}(V'_{j_1} + \mathrm{conv}\, A'_{j_1}) \cap \mathrm{int}(V'_{j_2} + \mathrm{conv}\, A'_{j_2}) = \emptyset \quad \forall j_1 \neq j_2 \in I_k.$$

Now the sets V_i $(i \in J_j)$ form a hierarchically affine partition of the set V'_j. Since the partition is given by affine functions, we may consider these functions on the whole L and take V''_i $(i \in J_j)$ such that $V_i = V''_i \cap V'_j$ and the sets V''_i $(i \in J_j)$ give a hierarchically affine partition of L.

We apply the inductive assumption to every family V''_i $(i \in J_j)$, its respective A'_j and the subset $\{l_i\}_{i\in J_j}$. So we obtain partitions of every A'_j into A''_i $(i \in J_j)$ such that $|A''_i| = l_i$, and in the case of Property 3

$$\bigcup_{i\in J_j} \cap_{a\in A''_i}(V''_i + a) = L,$$

or in the case of Property 4

$$\mathrm{int}(V''_{i_1} + \mathrm{conv}\, A''_{i_1}) \cap \mathrm{int}(V''_{i_2} + \mathrm{conv}\, A''_{i_2}) = \emptyset \quad \forall i_1 \neq i_2 \in J_j.$$

Thus in the case of Property 3

$$\begin{aligned}
&(\bigcap_{a\in A''_i} (V_i + a)) \bigcap (\bigcap_{a\in A'_j} (V'_j + a)) = \\
&\qquad = (\bigcap_{a\in A''_i} ((V_i + a) \cap (V'_j + a))) \bigcap (\bigcap_{a\in A'_j} (V'_j + a)) = \\
&\qquad\qquad = (\bigcap_{a\in A''_i} (V''_i + a)) \bigcap (\bigcap_{a\in A'_j} (V'_j + a)),
\end{aligned}$$

and therefore

$$\bigcup_{i\in J_j} \cap_{a\in A''_i}(V_i + a) \supseteq V'_j,$$

and

$$\bigcup_{i\in I_m} \cap_{a\in A''_i}(V_i + a) = L.$$

It means that Property 3 holds.

For Property 4 for indices $i_1 \neq i_2$ from the same J_j we have

$$\begin{aligned}
&\mathrm{int}(V_{i_1} + \mathrm{conv}\, A''_{i_1}) \cap \mathrm{int}(V_{i_2} + \mathrm{conv}\, A''_{i_2}) \subseteq \\
&\qquad \subseteq \mathrm{int}(V''_{i_1} + \mathrm{conv}\, A''_{i_1}) \cap \mathrm{int}(V''_{i_2} + \mathrm{conv}\, A''_{i_2}) = \emptyset,
\end{aligned}$$

and for indices $i_1 \neq i_2$ from J_{j_1} and J_{j_2} respectively

$$\operatorname{int}(V_{i_1} + \operatorname{conv} A''_{i_1}) \cap \operatorname{int}(V_{i_2} + \operatorname{conv} A''_{i_2}) \subseteq \\ \subseteq \operatorname{int}(V'_{j_1} + \operatorname{conv} A'_{j_1}) \cap \operatorname{int}(V'_{j_2} + \operatorname{conv} A'_{j_2}) = \emptyset.$$

In other words, for any $i_1 \neq i_2 \in I_m$

$$\operatorname{int}(V_{i_1} + \operatorname{conv} A''_{i_1}) \cap \operatorname{int}(V_{i_2} + \operatorname{conv} A''_{i_2}) = \emptyset.$$

Thus Property 4 holds. □

In the paper [1] a counterexample was given showing that even in the case of $\mathbb{R}^3$ there exist partitions that do not have Properties 1 or 2.

Still it makes sense to search for some more partitions with Properties 1 or 2. We shall give some variant of Conjectures 1 and 2 from [1], but before that we need a definition:

Definition. The partition of $\mathbb{R}^n$ into closed convex sets V_i $(i \in I_m)$ is called *ordered w.r.t. an ordered line* l (denote by $\leq_1$ the relation on l), if the family $\{V_i\}$ can be ordered by relation $\leq_2$ so that for any translate l' of l with translated relation $\leq_1$ we have $V_i \cap l' \leq_1 V_j \cap l'$ for all $V_i \leq_2 V_j$ $(i, j \in I_m)$.

Conjecture. *Any partition of* $\mathbb{R}^n$ *ordered w.r.t any ordered line, has Properties 1 and 2.*

As it was noted in [1] (Conjectures 3 and 4), the case of $n = 2$ of this conjecture is of interest itself because any partition of $\mathbb{R}^2$ is ordered w.r.t. any ordered line.

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