# PIERCING FAMILIES OF CONVEX SETS WITH $d$-INTERSECTION PROPERTY IN $\mathbb{R}^d$

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Abstract. In this paper we consider finite families of convex sets in $\mathbb{R}^d$ such that every $d$ or less sets of the family have a common point. For some families of this type we give upper bounds on the size of a finite set intersecting all sets of the family.

## 1. Introduction

We consider finite families of convex sets in $\mathbb{R}^d$ such that every $d$ or less sets have a common point. We are interested in the minimal size of a finite set in $\mathbb{R}^d$ having common point with every member of the family.

We begin with several definitions:

**Definition.** *$t$-transversal* for a family of sets $\mathcal{F}$ is a set $T$ of cardinality $t$ such that $\forall S \in \mathcal{F}\ S \cap T \neq \emptyset$.

**Definition.** Minimal $t$ such that a $t$-transversal of the family $\mathcal{F}$ exists is called *transversal number* or *piercing number* of $\mathcal{F}$, and denoted $\tau(\mathcal{F})$.

**Definition.** A family of sets $\mathcal{F}$ has *property* $\Pi_k$ if for any nonempty $\mathcal{G} \subseteq \mathcal{F}$ such that $|\mathcal{G}| \le k$ the intersection $\bigcap \mathcal{G}$ is not empty.

Helly's theorem states that $\Pi_{d+1}$ implies $\tau(\mathcal{F}) = 1$ for any finite family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$.

We can see that the case of $\Pi_d$ property in $\mathbb{R}^d$ is the closest to the case of Helly's theorem, and we may expect reasonable upper bounds for $\tau(\mathcal{F})$ here. Of course, the example of hyperplanes in general position shows that $\Pi_d$ property alone cannot guarantee any bound on $\tau(\mathcal{F})$. Hence we have to impose some limitations on family $\mathcal{F}$ to obtain some bounds for the piercing number. In this paper we mostly consider families of homothets or translates of some compact convex set.

This research was supported by the Russian Foundation for Basic Research grants No. 03-01-00801 and 06-01-00648, and by the President's of Russian Federation grant No. MK-5724.2006.1.

From here on we consider finite families of closed convex sets in $\mathbb{R}^d$ ($d \geq 2$) having property $\Pi_d$. We show that for for certain families $\Pi_d$ property implies $\tau(\mathcal{F}) \leq d+1$. We also give a linear in $d$ upper bound for the piercing number of families of euclidean balls in $\mathbb{R}^d$ with $\Pi_d$ property.

The simplest result of this type (Grünbaum's conjecture) was proved by the author in [4]:

**Theorem 1.** *Let $\mathcal{F}$ be a family of translates of a two-dimensional convex compact set. If $\mathcal{F}$ has property $\Pi_2$ then $\tau(\mathcal{F}) \leq 3$.*

In this paper we use the same main idea as in [4]. But the way of reasoning was quite cleared up, which lead to several more results.

**Theorem 2.** *Let $\mathcal{F}$ be a family of homothets of a centrally symmetric convex compact set in $\mathbb{R}^2$ and let any two sets in $\mathcal{F}$ be no more than two times different in size. If $\mathcal{F}$ has property $\Pi_2$ then $\tau(\mathcal{F}) \leq 3$.*

In [3] Grünbaum proved the upper bound $\tau(\mathcal{F}) \leq 7$ without any size constraint. Theorem 2 gives less piercing number with size constraint. The bound $\tau(\mathcal{F}) \leq 3$ in this theorem is tight, it is well-known that it is tight even for families of equal unit disks.

Using the same technique we prove another result of this kind for euclidean balls in $\mathbb{R}^d$:

**Theorem 3.** *Let $\mathcal{F}$ be a family of Euclidean balls in $\mathbb{R}^d$ with radii no more than $d$ times different. If $\mathcal{F}$ has property $\Pi_d$ then $\tau(\mathcal{F}) \leq d+1$.*

The existence of $(d+1)$-transversal for a family of equal balls with $\Pi_d$ property is proved in [1].

Theorem 3 generalizes this result for families of balls with size constraint. The author is not sure that the bound $\tau(\mathcal{F}) \leq d+1$ in Theorem 3 is tight.

Using this result we can give an upper bound on the piercing number of a family of balls in $\mathbb{R}^d$ with $\Pi_d$ property without any size constraint:

**Theorem 4.** *Let $\mathcal{F}$ be a family of Euclidean balls in $\mathbb{R}^d$. If $\mathcal{F}$ has property $\Pi_d$ then $\tau(\mathcal{F}) \leq 3(d+1)$ when $d \geq 5$ and $\tau(\mathcal{F}) \leq 4(d+1)$ when $d \leq 4$.*

It seems that the bound in this theorem can be improved, especially for small $d$. In case of $d = 2$ Theorem 4 gives $\tau(\mathcal{F}) \leq 12$, while Danzer's result (see [1, 2]) gives $\tau(\mathcal{F}) \leq 4$.

For a family of positive homothets of a simplex in $\mathbb{R}^d$ we have bound $d+1$ without size constraints:

**Theorem 5.** *Let $\mathcal{F}$ be a family of positive homothets of a simplex in $\mathbb{R}^d$. If $\mathcal{F}$ has property $\Pi_d$ then $\tau(\mathcal{F}) \leq d+1$.*

The author does not know whether Theorem 5 gives tight upper bound on the piercing number for $d > 2$. For $d = 2$ the bound is tight even for families of equal triangles. In this case the piercing problem is equivalent to the following covering problem: if every two points of a closed set $S \subset \mathbb{R}^2$ can be covered by a translate of triangle $T$, then $S$ can be covered by 3 translates of $T$. Taking $S = -T$ we can see that 3 translates are necessary. Here the set $S$ is infinite, but by the standard compactness reasoning some its finite subset still needs 3 translates of $T$ to be covered.

Theorem 3 can be generalized to the case when the sets in the family do not have to be homothets of each other. We need some definitions to formulate the result.

**Definition.** A convex compact set in $\mathbb{R}^d$ is called *$R$-upper convex* if it is an intersection of balls of radius $R$.

**Definition.** A convex compact set in $\mathbb{R}^d$ is called *$R$-lower convex* if it is a union of balls of radius $R$.

**Theorem 6.** *Let $\mathcal{F}$ be a family of convex compact sets in $\mathbb{R}^d$. Let every set in $\mathcal{F}$ be $R$-lower convex and $dR$-upper convex for some constant $R > 0$. If $\mathcal{F}$ has property $\Pi_d$ then $\tau(\mathcal{F}) \leq d+1$.*

The following question remains open: whether the upper bound $\tau(\mathcal{F}) \leq d+1$ (or some other linear in $d$ bound) is true for families of translates of a convex compact set.

## 2. Some consequences of $\Pi_d$ property

To prove theorems in this paper we explicitly construct a $(d+1)$-element set and prove that this is a transversal for $\mathcal{F}$.

This construction only uses $\Pi_d$ property of some family $\mathcal{F}$ of convex closed sets in $\mathbb{R}^d$, but it does not give a transversal of $\mathcal{F}$ in the general case.

First we define a family of halfspaces that can test the non-existence of a common point of a family $\mathcal{F}$ of compact convex sets.

**Definition.** Let $\{K_1, K_2, \dots, K_{d+1}\} \subseteq \mathcal{F}$. A family of halfspaces

$$\{H_1, H_2, \dots, H_{d+1}\},$$

where for any $i = 1, \dots, d+1$ $K_i \subseteq H_i$ is called *test family of halfspaces* for family $\mathcal{F}$.

**Definition.** If a test family of halfspaces for $\mathcal{F}$ has an empty intersection we call it *non-intersecting* test family of halfspaces.

We need some lemmas:

**Lemma 1.** *Let $\mathcal{F}$ have $\Pi_d$ property. Then for every non-intersecting test family of halfspaces $\{H_1, H_2, \dots, H_{d+1}\}$ for $\mathcal{F}$ the set*

$$S(H_1, H_2, \dots, H_{d+1}) = \operatorname{cl}\left(\mathbb{R}^d \setminus \bigcup_{i=1}^{d+1} H_i\right)$$

*is a simplex.*

*Proof.* The family $\mathcal{H} = \{H_1, H_2, \dots, H_{d+1}\}$ has empty intersection while every its proper subfamily have a common point. Let for $i = 1, \dots, d+1$

$$x_i \in \bigcap_{j \neq i} H_j.$$

If the points $\{x_i\}_{i=1}^{d+1}$ does not make a simplex then they lie in a $d-1$-dimensional hyperplane and by the Radon's theorem the set of indices $[d+1]$ can be partitioned into $I_1$ and $I_2$ so that there exists

$$x \in \operatorname{conv}\{x_i\}_{i \in I_1} \cap \operatorname{conv}\{x_i\}_{i \in I_2},$$

but in this case $x \in \bigcap \mathcal{H}$.

Hence $\{x_i\}_{i=1}^{d+1}$ make a simplex $S$ and halfspaces $H_i$ contain its respective facets. If the family $\mathcal{H}$ cover $S$ then by Sperner's lemma $\bigcap \mathcal{H} \cap S \neq \emptyset$, which is not true. Thus we have a point

$$x \in \operatorname{int} S \quad \forall i = 1, \dots, d+1 \ x \notin H_i.$$

Consider some $y \in \mathbb{R}^d \setminus S$. If $y$ is not contained in any halfspace of $\mathcal{H}$ then the segment $[x, y]$ does not intersect any halfspace of $\mathcal{H}$ (this is true because the complement to a halfspace is convex). Hence $z = [x, y] \cap \operatorname{bd} S$ is not contained in any of $\mathcal{H}$, This is a contradiction with the fact that $\operatorname{bd} S$ is covered by $\mathcal{H}$.

So we have $\mathbb{R}^d \setminus S \subseteq \bigcup \mathcal{H}$ and $S(H_1, H_2, \dots, H_{d+1}) = \operatorname{cl}\left(\mathbb{R}^d \setminus \bigcup_{i=1}^{d+1} H_i\right)$ is a polytope with nonempty interior and $d+1$ facets, so it has to be a $d$-dimensional simplex. □

**Lemma 2.** *If a family $\mathcal{F}$ with $\Pi_d$ property does not have non-intersecting test family of halfspaces then $\bigcap \mathcal{F} \neq \emptyset$.*

*Proof.* If $|\mathcal{F}| \leq d$ the statement is obvious.

Otherwise, by Helly's theorem we find $\mathcal{K} = \{K_1, K_2, \dots, K_{d+1}\} \subseteq \mathcal{F}$ such that $\bigcap \mathcal{K} = \emptyset$.

Let for $i = 1, \dots, d+1$ $L_i$ denote the set of linear functions $l(x)$ such that $l(x) \le 0$ for all $x \in K_i$. By the standard properties of convex duality, if $\bigcap \mathcal{K} = \emptyset$ then the constant function $l(x) \equiv 1 \in \operatorname{conv} \bigcup_{i=1}^{d+1} L_i$. Equivalently, there are non-negative numbers $\{a_i\}_{i=1}^{d+1}$ and functions $\{l_i\}_{i=1}^{d+1}$ ($l_i \in L_i$ $\forall i = 1, \dots, d+1$) such that

$$1 = \sum_{i=1}^{d+1} a_i l_i(x).$$

Denote $H_i = \{x : l_i(x) \le 0\}$. These sets have no common point since in the common point the above equality cannot hold. For all $i = 1, \dots, d+1$ $K_i \subseteq H_i$ and each $H_i$ is either a halfspace or coincides with the whole $\mathbb{R}^d$. If we substitute those $H_i$'s that are equal to $\mathbb{R}^d$ by arbitrary halfspaces containing respective $K_i$'s, we obtain a non-intersecting test family of halfspaces. □

**Lemma 3.** *Let the family $\mathcal{F}$ have $\Pi_d$ property. Then either $\tau(\mathcal{F}) = 1$ or there exists such a non-intersecting test family of halfspaces $\{H_1, H_2, \dots, H_{d+1}\}$ that the volume of $S(H_1, H_2, \dots, H_{d+1})$ is maximal over all non-intersecting test families of halfspaces.*

*Proof.* If there is no non-intersecting test family of halfspaces then by Lemma 2 $\tau(\mathcal{F}) = 1$.

In the other case it is sufficient to show that for every fixed subfamily

$$\mathcal{K} = \{K_1, K_2, \dots, K_{d+1}\} \subseteq \mathcal{F}$$

with empty intersection the maximum of $\operatorname{vol} S(H_1, H_2, \dots, H_{d+1})$ over all non-intersecting test families $\mathcal{H} = \{H_1, H_2, \dots, H_{d+1}\}$ corresponding to $\mathcal{K}$ is reached. This is sufficient to prove the lemma since the number of subfamilies $\mathcal{K}$ is finite.

Any $d$ sets of family $\mathcal{K}$ have nonempty intersection, so we find for all $i = 1, \dots, d+1$

$$x_i \in \bigcap_{j \ne i} K_i.$$

As in the proof of Lemma 1

$$S(H_1, H_2, \dots, H_{d+1}) \subseteq \operatorname{conv}\{x_1, x_2, \dots, x_{d+1}\},$$

hence $\operatorname{vol}(S(H_1, H_2, \dots, H_{d+1}))$ is bounded.

It is clear that if we take instead of $H_i$ the unique inner support halfspace for $K_i$ contained in $H_i$ the volume of $S(H_1, H_2, \dots, H_{d+1})$ becomes larger. Every support halfspace of $K_i$ is determined by its unit normal vector $n_i$, so the variety of the families of support halfspaces to the sets of $\mathcal{K}$ has the topology of a cartesian product of $d+1$ unit spheres. We consider the subset $X$ of this space that corresponds to non-intersecting test families for $\mathcal{K}$. The vertices of $S(H_1, H_2, \dots, H_{d+1})$ and its volume are continuous functions on $X$.

Let $\sup_X \operatorname{vol}(S(H_1, H_2, \dots, H_{d+1})) = 2\varepsilon > 0$. Then we can easily check that the variety of non-intersecting test families $\mathcal{H} = \{H_1, H_2, \dots, H_{d+1}\}$ such that

$$\operatorname{vol}(S(H_1, H_2, \dots, H_{d+1})) \geq \varepsilon$$

is closed, hence the volume takes its maximum value $2\varepsilon$ at some point of $X$. □

The above lemmas allow us to make a definition:

**Definition.** Let $\mathcal{F}$ be a finite family of convex closed sets in $\mathbb{R}^d$ with $\Pi_d$ property and let $\bigcap \mathcal{F} = \emptyset$. Denote the simplex $S(H_1, H_2, \dots, H_{d+1})$ with maximal volume for some non-intersecting test family of halfspaces $\{H_1, H_2, \dots, H_{d+1}\}$ by $S(\mathcal{F})$. Denote the convex hull of the mass centers of facets of $S(\mathcal{F})$ by $s(\mathcal{F})$.

The importance of these simplices is explained by the following lemma:

**Lemma 4.** *Let $\mathcal{F}$ be a finite family of convex closed sets in $\mathbb{R}^d$ with $\Pi_d$ property having no common point. In this case for any $S \in \mathcal{F}$ $S \cap s(\mathcal{F}) \neq \emptyset$.*

To prove Lemma 4 we need another lemma:

**Lemma 5.** *Let $S$ be a simplex in $\mathbb{R}^d$ with vertices $\{v_1, v_2, \dots, v_{d+1}\}$, let $S'$ be a simplex with vertices $\{v'_1, v'_2, \dots, v'_d, v_{d+1}\}$. Let points $v'_i$ $(i = 1, \dots, d)$ lie on the rays $v_{d+1}v_i$ respectively. If $\operatorname{int} S'$ contains the mass center of $\{v_1, \dots, v_d\}$ then $\operatorname{vol} S' > \operatorname{vol} S$.*

*Proof.* Take a coordinate frame with origin $v_{d+1}$ and base

$$\{v'_1 - v_{d+1}, v'_2 - v_{d+1}, \dots, v'_d - v_{d+1}\},$$

let $i$-th coordinate of $v_i$ be $x_i$. If $\operatorname{int} S'$ contains the mass center of $\{v_1, \dots, v_d\}$ then

$$\sum_{i=1}^{d} x_i < d,$$

and by the mean inequality $\prod_{i=1}^d x_i < 1$, which implies $\operatorname{vol} S' > \operatorname{vol} S$. □

*Proof of Lemma 4.* Consider the subfamily

$$\mathcal{K} = \{K_1, K_2, \dots, K_{d+1}\} \subseteq \mathcal{F},$$

corresponding to the non-intersecting test family of halfspaces

$$\mathcal{H} = \{H_1, H_2, \dots, H_{d+1}\},$$

where $S(\mathcal{F}) = S(H_1, H_2, \dots, H_{d+1})$. Denote the respective vertices of $S(\mathcal{F})$ and $s(\mathcal{F})$ by $\{v_1, v_2, \dots, v_{d+1}\}$ and $\{w_1, w_2, \dots, w_{d+1}\}$.

The volume of $S(\mathcal{F})$ is maximal, so each $K_i$ has to touch its respective facet of $S(\mathcal{F})$.

Assume the contrary: $K_i$ does not intersect $s(\mathcal{F})$.

Let $C_i$ be a cone with vertex $w_i$ spanned by rays $w_i w_j$ $(j \neq i)$. $K_i$ cannot intersect $C_i$ because these sets are separated by $\operatorname{bd} H_i$ and could only have intersection in $w_i \in s(\mathcal{F})$.

Then there exists a halfspace $H_i' \supseteq K_i$ such that $H_i' \cap C_i = \emptyset$. The family of halfspaces $(\mathcal{H} \setminus \{H_i\}) \cup \{H_i'\}$ is non-intersecting $(\bigcap_{j \neq i} H_i \subseteq C_i)$ and is a test family. By Lemma 5

$$\operatorname{vol} S(H_1, \dots, H_i', \dots, H_{d+1}) > \operatorname{vol} S(\mathcal{F})$$

and we have a contradiction.

Now take $K \in \mathcal{F} \setminus \mathcal{K}$. Assume that $K \cap s(\mathcal{F}) = \emptyset$.

There exists a halfspace $H \supseteq K$ that does not intersect $s(\mathcal{F})$. $H \not\supseteq S(\mathcal{F})$ and we find the farthest from $H$ vertex of $S(\mathcal{F})$ and denote it $v_i$.

Then $\bigcap_{j \neq i} H_j$ is a cone spanned by the rays that has their origins at $v_i$ and do not intersect $H$. Hence $H \cap \bigcap_{j \neq i} H_j = \emptyset$ and the family $(\mathcal{H} \setminus \{H_i\}) \cup \{H\}$ has no common point. Considering simplices $S((\mathcal{H} \setminus \{H_i\}) \cup \{H\})$ and $S(\mathcal{F})$ we again come to contradiction by Lemma 5. □

We need another lemma to prove Theorem 4:

**Lemma 6.** *Suppose we have a simplex $S$ in $\mathbb{R}^d$ $(d \geq 5)$, its outer halfspaces corresponding to facets being $\mathcal{H} = \{H_1, H_2, \dots, H_{d+1}\}$. Let a ball $B$ of radius $R$ together with the family $\mathcal{H}$ give a family with $\Pi_d$ property. Then the radius $r$ of the ball inscribed in $S$ satisfies the inequality*

$$r \leq \frac{1}{\lfloor \frac{d+1}{2} \rfloor - 1} R \leq 1/2R.$$

*Proof.* Let the vertices of $S$ be $\{v_1, v_2, \dots, v_{d+1}\}$. Consider the orthogonal projection $p : \mathbb{R}^d \to V$ taking pairs $\{v_1, v_2\}$, $\{v_3, v_4\}$, and so on to single points. If $d+1$ is odd let $p$ take $v_{d-1}, v_d, v_{d+1}$ into one point, otherwise, $p$ takes pairs into single points. We have a simplex $S' = p(S)$ with $d' + 1 = \lfloor \frac{d+1}{2} \rfloor$ vertices, let its outer halfspaces of facets be $\mathcal{H}' = \{H_1', H_2', \dots, H_{d'+1}'\}$. We also have a ball $B' = p(B)$.

In the barycentric coordinates $\{x_1, x_2, \dots, x_{d+1}\}$ of $S$ and $\{y_1, y_2, \dots, y_{d'+1}\}$ of $S'$ the map $p$ is given by:

$$y_1 = x_1 + x_2$$
$$y_2 = x_2 + x_3$$
$$\dots$$
$$y_{d'+1} = x_d + x_{d+1} \quad \text{or} \quad y_{d'+1} = x_{d-1} + x_d + x_{d+1}.$$

It is easy to see that $B'$ has a common point with every family $\mathcal{H}' \setminus \{H'_i\}$. These common points form a simplex containing $S'$, hence $B'$ contains $S'$.

Then we make a homothety with center at the mass center of $S'$ and scale ratio $-\dfrac{1}{d'}$. It takes $B'$ to the ball $B''$ of radius $\frac{1}{d'}R$ that intersects all facets of $S'$. So the radius of the inscribed ball of $S'$ cannot be larger than $\frac{1}{d'}R$, since the inscribed ball is the ball of minimal radius intersecting all the facets of the simplex.

Hence the radius of the inscribed ball of $S$ cannot be larger than $\frac{1}{d'}R$ too. □

## 3. Concave simplices of curvature radius $R$

We give some definitions:

**Definition.** Let a family$\{B_1, B_2, \dots, B_{d+1}\}$ of euclidean balls of radius $R$ and centers $\{o_1, o_2, \dots, o_{d+1}\}$ in $\mathbb{R}^d$ have the following property: every $d$ balls have a common point while the whole family has no common point. In this case we call the set

$$\operatorname{cl}\left(\operatorname{conv}\{o_1, o_2, \dots, o_{d+1}\} \setminus \bigcup_{i=1}^{d+1} B_i\right)$$

a *concave simplex of curvature radius $R$.*

In the plane we consider arbitrary norm $\|\cdot\|$ and give a non-euclidean version of the above definition:

**Definition.** Let a family $\{B_1, B_2, B_3\}$ of balls of radius $R$ and centers $\{o_1, o_2, o_3\}$ w.r.t. norm $\|\cdot\|$ in the plane have the following property: every 2 balls intersect while the whole family has no common point. The set

$$\operatorname{cl}\left(\operatorname{conv}\{o_1, o_2, o_3\} \setminus \bigcup_{i=1}^{3} B_i\right)$$

is called a *concave simplex of curvature radius $R$* in norm $\|\cdot\|$.

By Sperner's lemma the simplices defined above are nonempty, otherwise the family should have a common point.

We could define a concave simplex for arbitrary norm in any dimension, but in general they do not have the properties we want because the bounded connected component of $\mathbb{R}^d \setminus \bigcup_{i=1}^{d+1} B_i$ does not have to lie in $\operatorname{conv}\{o_1, o_2, \dots, o_{d+1}\}$ for $d > 2$.

We prove several lemmas about concave simplices of curvature radius $R$. Unless specially noted, our reasoning is valid for both the cases of an arbitrary two-dimensional norm and the euclidean norm in $\mathbb{R}^d$.

**Definition.** Let $T$ be a concave simplex in $\mathbb{R}^d$. A *vertex* of $T$ is a point on its boundary that belongs to boundaries of exactly $d$ balls that determine $T$.

**Lemma 7.** *Let $T$ be a concave simplex of curvature radius $R$. Then*
*1) $\operatorname{int} T$ does not intersect the boundary of $S = \operatorname{conv}\{o_1, o_2, \dots, o_{d+1}\}$;*
*2) $T$ is contained in the convex hull of its vertices;*
*3) the number of vertices of $T$ is $d+1$.*

*Proof.* The first statement can be deduced from the fact that each facet of $S$ is contained in a union of balls of radius $R$ with centers in the vertices of the facet.

Each extremal point of $T$ has to be its vertex, hence $T$ is contained in the convex hull of its vertices.

Each vertex of $T$ corresponds to the intersection of $d$ spheres of radius $R$. Centers of these spheres make a facet $F$ of $S$. If the spheres have two intersection points then their intersection points lie on different sides of the hyperplane of $F$, so only one of the points can be a vertex of $T$. In the case of the plane, when the balls are not strictly convex, the intersection of the spheres can be a line segment. But in this case a vertex of $T$ is necessarily in the end of the segment and one of the ends is on the other side of $F$.

Finally we have no more than $d+1$ vertices of $T$, their number cannot be less because $\operatorname{int} T \neq \emptyset$. ☐

**Lemma 8.** *A concave simplex $T$ of curvature radius $R$ in two-dimensional case has diameter no more than $R$ in the norm $\|\cdot\|$; in $d$-dimensional case it has diameter no more than $\frac{2}{\sqrt{d^2-1}}R < R$.*

*Proof.* In two-dimensional case $T$ lies in the triangle formed by midpoints of sides of triangle $o_1o_2o_3$, therefore its diameter does not exceed $R$.

In the case of euclidean balls consider the point $o$ equidistant from $o_1, o_2, \dots, o_{d+1}$. It is not contained in any of the balls $B_{o_i}(R)$ and is contained in $\operatorname{conv}\{o_1, o_2, \dots, o_{d+1}\}$, hence, $o \in T$. The result of Danzer (see Theorem 6.8 in [1]) implies that $|oo_i| \le \frac{d}{\sqrt{d^2-1}}R$.

The length of any tangent from $o$ to the ball $B_{o_i}(R)$ does not exceed $\frac{1}{\sqrt{d^2-1}}R$ and any line segment from $o$ to a point on $\operatorname{bd} B_{o_i}(R)$ that does not intersect $\operatorname{int} B_{o_i}(R)$ has length no more than $\frac{1}{\sqrt{d^2-1}}R$.

Every vertex $v$ of $T$ lies on the radical axis of some $d$ balls, this radical axis passes through $o$, thus all segments $ov$ lie in $T$ and do not intersect $\operatorname{int} B_{o_i}(R)$. Hence all segments $ov$ have length no more than $\frac{1}{\sqrt{d^2-1}}R$, which implies the statement of the lemma. □

**Lemma 9.** *A ball of radius $R' \ge R$ intersects a concave simplex $T$ of curvature radius $R$ iff it contains some vertex of $T$.*

*Proof.* Consider all the balls of radius $R'$ that intersect $T$. Denote the set of their centers by $X$.

Consider all the balls of radius $R'$ that contain some vertex of $T$. Denote the set of their centers by $Y$. It is clear to see that

$$Y = \bigcup_{i=1}^{d+1} B_{o_i}(R').$$

By Lemma 8 $\operatorname{diam} T \le R$ and each ball $B_{o_i}(R')$ contains $T$. Thus $Y$ is star-shaped with any center in $T$. Similarly, $X$ is star-shaped with any center in $T$. To prove the inclusion $X \subseteq Y$ we only have to prove that $\operatorname{bd} X \subseteq Y$.

Let $x \in \operatorname{bd} X$ and $B = B_x(R')$. Clearly $B \cap \operatorname{int} T = \emptyset$ and $B \cap \operatorname{bd} T \ne \emptyset$.

Consider the case of the euclidean norm first.

Let $y \in \operatorname{bd} X \cap B$. If $y$ is not a vertex of $T$ then it lies on at most $d-1$ spheres of radius $R$ that determine $T$. Let their centers be $o_1, \dots, o_k$, $k \le d-1$. Considering the linear approximation of $T$ and $B$ in some vicinity of $y$ we see that $\operatorname{int} X \cap \operatorname{int} B = \emptyset$ implies that

$$x - y = \sum_{i=1}^{k} \alpha_i (o_i - y) \quad \forall\ i = 1, \dots, k\ \alpha_i \ge 0.$$

If there is only one index $i$ such that $\alpha_i \ne 0$ then for such $i$ we have $B \supseteq B_{o_i}(R)$, in this case $B$ contains at least $d$ vertices of $T$ that lie on $\operatorname{bd} B_{o_i}(R)$.

Otherwise, consider the affine subspace

$$L = \{p \in \mathbb{R}^d : |p - o_1| = |p - o_2| = \dots = |p - o_k|\}.$$

We have $\dim L \ge 2$, $y \in L$. All intersections $L \cap B_{o_i}(R)$ $(i = 1, \dots, k)$ are the same ball $B' \subset L$ and $L \cap B$ is another ball $B'' \subset L$.

The direction of vector $x-y$ does not coincide with any of $o_i - y$ but lies in their cone hull. All the angles between $o_i - y$ and $L$ are equal to some $\alpha$, so the angle between $x - y$ and $L$ is less than $\alpha$, hence the radius of $B'$ is less than the radius of $B''$.

The intersection $\operatorname{int} T \cap L$ in some vicinity of $y$ coincides with $L \setminus B'$ and $B''$ intersects $L \setminus B'$ in any vicinity of $y$, this is a contradiction with $B \cap \operatorname{int} T = \emptyset$.

The case of $d = 2$ and arbitrary norm with smooth and strictly convex ball is made in the same manner as above, in this case $k = 1$ and we do not have to use equidistant set (which is not an affine subspace in general). Then we can approximate any norm by a norm with smooth and strictly convex ball and proceed by going to a limit. □

**Lemma 10.** *$R$-lower convex set intersects a concave simplex $T$ of curvature radius $R$ iff it contains a vertex of $T$.*

*Proof.* Trivially deduced from Lemma 9. □

## 4. Proofs of the theorems

*Proof of Theorems 2 and 3.* If $\tau(\mathcal{F}) = 1$ the statement is true. Otherwise by Lemma 3 we have $S = S(\mathcal{F})$ and $s = s(\mathcal{F})$. We call elements of $\mathcal{F}$ balls.

Denote the maximal radius of the ball in $\mathcal{F}$ by $R$.

Take the balls $K_1,K_2,\dots,K_{d+1}$ corresponding to $S$, they touch the facets of $S$ in the respective vertices of $s$. Consider the balls $K'_1,K'_2,\dots,K'_{d+1}$ with radius $R$ touching the respective facets of $S$ in vertices of $s$.

It can be easily seen that $K'_i \supseteq K_i$ for all $i = 1,\dots,d+1$. Each $d$ of $K'_i$ have a common point and they all do not have a common point, hence they define a concave simplex $T$ of curvature radius $R$, obviously $T \supseteq S$. Let the homothety with scale ratio $-1/d$ that takes $S$ to $s$ take $T$ to $T'$, the latter concave simplex has curvature radius $R/d$.

By Lemma 4 any $K \in \mathcal{F}$ intersects $s$ and therefore intersects $T'$, its radius being at least $R/d$. Hence by Lemma 9 $K$ contains one of $d+1$ vertices of $T'$. □

*Proof of Theorem 4.* If $\mathcal{F}$ has a common point it is nothing to prove.

Otherwise, consider the ball $B(\mathcal{F})$ of minimal radius $r$ intersecting every ball in $\mathcal{F}$. This is a well-known consequence of Helly's theorem that this ball is the ball of minimal radius intersecting $d+1$ balls of some subfamily $\mathcal{F}' = \{B_1, B_2, \dots, B_{d+1}\} \subseteq \mathcal{F}$ (remember that $\mathcal{F}$ has $\Pi_d$ property) and each intersection $B(\mathcal{F}) \cap B_i$ is exactly one point $v_i$.

Let $H_i$ be the outer support halfspace for $B(\mathcal{F})$ in the point $v_i$. Then $S = \mathbb{R}^d \setminus \bigcup_{i=1}^{d+1} \operatorname{int} H_i$ is a simplex with inscribed ball $B(\mathcal{F})$.

Let the ball $B \in \mathcal{F}$ have minimal radius $R$. Consider the case $d \geq 5$ first. Obviously, $B$ and $S$ satisfy the conditions of Lemma 6 and we have $r \leq 1/2R$.

Note that a ball of radius $r$ can be put into a concave simplex of curvature radius

$$R' = \frac{1}{\frac{d}{\sqrt{d^2-1}} - 1} r = (d\sqrt{d^2-1} + d^2 - 1)r \leq 2d^2 r \leq d^2 R.$$

Thus the subfamily of balls in $\mathcal{F}$ with radii $\geq d^2 R$ has a transversal of cardinality $d+1$, which is the set of vertices of this concave simplex. Other balls in $\mathcal{F}$ can be partitioned into two subfamilies with radii from $R$ to $dR$ and from $dR$ to $d^2R$, by Theorem 3 these subfamilies have $(d+1)$-transversals. In total we obtain a $3(d+1)$-transversal for $\mathcal{F}$.

In the case $d \leq 4$ we note that all balls in $\mathcal{F}$ intersect the smallest ball in $\mathcal{F}$ of radius $R$, the latter can be put into a concave simplex of curvature radius

$$(d\sqrt{d^2-1} + d^2 - 1)R \leq d^3 R.$$

The vertices of this concave simplex make a transversal for balls in $\mathcal{F}$ with radii $\geq d^3R$. Other balls can be partitioned into three families with $(d+1)$-transversals to obtain the transversal for $\mathcal{F}$ of size $4(d+1)$. $\square$

*Proof of Theorem 5.* We omit the obvious case $\tau(\mathcal{F}) = 1$ and consider $S = S(\mathcal{F})$ and $s = s(\mathcal{F})$.

Note that any simplex $K \in \mathcal{F}$ is a negative homothet of $S$ and a positive homothet of $s$. This can be deduced from considering the sets of outer and inner normals to facets of $S$, $K$, and $s$.

Assume that $K$ does not contain any vertex of $s$.

If $K$ intersects some $d$ facets of $s$ it has to contain their intersection, which is a vertex of $s$. Hence there are two facets of $s$ that $K$ does not intersect. Let these facets correspond to the halfspaces $H_1$ and $H_2$ from the family $\{H_1, H_2, \dots, H_{d+1}\}$ of outer support halfspaces for facets of $S$.

Denote the translate of halfspace $H_i$ containing the respective facet of $s$ in its boundary by $G_i$.

$K$ is a homothet of $s$, intersects $s$, and does not intersect the facet $\operatorname{bd} G_1 \cap s$. Hence $K \subseteq G_1$. Similarly $K \subseteq G_2$. We show that the intersection

$$G_1 \cap G_2 \cap H_3 \cap \ldots \cap H_{d+1}$$

is either empty or consists of one vertex of $s$, the latter is possible when $d = 2$ only. For all $i = 1, \dots, d+1$ $H_i \subseteq G_i$ and therefore

$$G_1 \cap G_2 \cap H_3 \cap \dots \cap H_{d+1} \subseteq G_1 \cap G_2 \cap G_3 \cap \dots \cap G_{d+1} = s.$$

The simplex $s$ intersects halfspaces $H_3, \dots, H_{d+1}$ at one point each, hence

$$G_1 \cap G_2 \cap H_3 \cap \dots \cap H_{d+1}$$

can be non-empty when $d = 2$ only, in this case in consists of one vertex of $s$.

Note that $K$ does not contain any vertex of $s$ and

$$K \cap \bigcap_{i=3}^{d+1} K_i \subseteq K \cap \bigcap_{i=3}^{d+1} H_i = \emptyset,$$

which is a contradiction with $\Pi_d$ property. □

We need another lemma to prove theorem 6.

**Lemma 11.** *If a halfspace $H$ contains an $R$-upper convex set $K$ then there exists a ball $B$ of radius $R$ such that $K \subseteq B \subseteq H$.*

The author cannot give the exact reference to the first proof of this fact, a proof of this fact can be found in [5].

*Proof of Theorem 6.* We omit the case $\tau(\mathcal{F}) = 1$ and consider $S = S(\mathcal{F})$ and $s = s(\mathcal{F})$.

Take the sets $K_1,K_2,\dots,K_{d+1}$ in $\mathcal{F}$ that correspond to $S$ and consider the balls $K'_1,K'_2,\dots,K'_{d+1}$ with the following properties: the radius of $K'_i$ is $dR$, $K'_i \supseteq K_i$, and $K'_i \cap S = K_i \cap S$. Such $K'_i$ can be found by Lemma 11 if we consider for each $K_i$ its respective outer support halfspace $H_i$ for $S$.

The balls $K'_1,K'_2,\dots,K'_{d+1}$ have $\Pi_d$ property and have no common point, so they make a concave simplex $T$ of curvature radius $dR$ and $T \supseteq S$. Let the homothety that takes $S$ to $s$ take $T$ to $T'$, the latter has curvature radius $R$.

By Lemma 4 any $K \in \mathcal{F}$ intersects $s$ and therefore intersects $T'$. By Lemma 10 $K$ contains one of $d+1$ vertices of $T'$. □

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