The 5$^{\text{th}}$ German-Russian Week of the Young Researcher

"Discrete Geometry"

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Contents

Log-concavity of Whitney numbers of the first kind

Karim Adiprasito

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(joint work with June Huh and Eric Katz)

A conjecture of Read predicts that the coefficients of the chromatic polynomial of any graph form a log-concave sequence. A related conjecture of Welsh predicts that the number of linearly independent subsets of varying sizes form a log-concave sequence for any configuration of vectors in a vector space.

All known proofs use Hodge theory for projective varieties, and the more general conjecture of Rota for possibly "nonrealizable" configurations/matroids is still open, mainly because no algebraic variety is available to use intersection theory on. In my talk, I will present a complete solution to Rota's conjecture relying on a purely combinatorial proof.

A Quantitative Doignon–Bell–Scarf Theorem

Iskander Aliev

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(joint work with Robert Bassett, Jesus De Loera, and Quentin Louveaux)

The famous Doignon–Bell–Scarf theorem is a Helly-type result about the existence of integer solutions to systems of linear inequalities. The purpose of this talk is to present the following quantitative generalization:

Given an integer k, we prove that there exists a constant $c(n, k)$, depending only on the dimension n and k, such that if a bounded polyhedron

$$\{x \in \mathbb{R}^n : Ax \leq b\}$$

contains exactly k integer points, then there exists a subset of the rows, of cardinality no more than $c(n, k)$, defining a polyhedron that contains exactly the same k integer points.

In this case $c(n, 0) = 2^n$ as in the original case of Doignon–Bell–Scarf for infeasible systems of inequalities. We present new upper and lower bounds for the constant $c(n, k)$ and discuss some corollaries of the obtained results.

Small subset sums

Gergely Ambrus

Alfréd Rényi Institute of Mathematics, Budapest

(joint work with Imre Bárány and Victor Grinberg)

Consider a real d-dimensional normed space. Let V be a set of n vectors of norm at most 1, which add up to 0. We prove that for every $k \leq n$, there exists a subset U of V with exactly k elements, whose sum has norm at most $\lceil d/2 \rceil$. We also demonstrate that for general norms, this bound is the best possible. For the Euclidean and the maximum norms, we strengthen the above estimate to $O(\sqrt{d})$.

The chromatic numbers of metric spaces with several forbidden distances

Alexei Berdnikov

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The chromatic number of a metric space (X, ρ) with a set of *forbidden distances* $\mathcal{A}$ is the smallest number $\chi((X, \rho), \mathcal{A})$ of colors needed to color all the points in X in such a way that any two points at a distance $a \in \mathcal{A}$ receive different colors. The problem of finding the chromatic number of the Euclidean space with one forbidden distance was proposed by Nelson in 1950 and is now one of the most important problems of combinatorial geometry.

In our talk, we will present new lower bounds for the chromatic numbers of $(\mathbb{R}^n, \ell_p)$ with k forbidden distances and an arbitrary $p \in \mathbb{N}$.

Cutting, embedding, bouncing characteristic classes

Pavle Blagojević

Freie Universität Berlin

(joint work with Imre Bárány, Frederick Cohen, Wolfgang Lück, Roman Karasev, András Szűcs, and Günter M. Ziegler)

The properties of the regular representation bundles over the configuration space of k distinct points in the Euclidean space has classically been studied extensively by F. Cohen, R. Cohen, Chisholm, Handel, Kuhn, Neisendorfer, V. Vassiliev, and many others.

Motivated by geometric problems we present new computations of twisted Euler classes, Stiefel–Whitney classes and their monomials as well as corresponding Chern classes of these bundles.

Thus, we not only extend and complete previous work, supplying for example a proof for a conjecture by Vassiliev, but also make progress in solving and extending variety of problems from Discrete Geometry, among them

1. the conjecture by Nandakumar and Ramana Rao that every convex polygon can be partitioned into k convex parts of equal area and perimeter;

2. Borsuk's problem on the existence of "k-regular maps" between Euclidean spaces, which are required to map any k distinct points to k linearly independent vectors;

3. Ghomi and Tabachnikov problem about the existence of "ℓ-skew smooth embeddings" from a smooth manifold M to a Euclidean space E, which are required to map tangent spaces at ℓ distinct points of M into ℓ skew subspaces of E.

Sections of the regular simplex – Volume formulas and estimates

Hauke Dirksen

Kiel University

We will state a general formula to compute the volume of the section of the regular n-simplex with some k-dimensional subspace. For hyperplane sections close to the centroid we give the optimal upper bound using this formula.

S. Webb considered *central* sections of the simplex. He derived a formula and showed that the hyperplane through the centroid containing $n-1$ vertices gives the maximal volume.

We generalize the formula to arbitrary dimensional sections that do not necessarily have to contain the centroid. Then we show that, for prescribed small distance of a hyperplane to the centroid, still the hyperplane containing $n-1$ vertices is volume maximizing. The proof also yields a new and short argument for Webb's result.

[1] K. Ball. Cube slicing in $\mathbb{R}^n$. *Proc. Amer. Math. Soc.* 97:3 (1986), 465–473.

[2] S. Webb. Central slices of the regular simplex. *Geometriae Dedicata* 61:1 (1996), 19–28.

Enumerating simplicial polytopes

Moritz Firsching

Freie Universität Berlin

The classification of polytopes has been studied since antiquity. Since the 1980s, no significant progress has been made in the classification of simplicial polytopes with few vertices in low dimensions. Recently we were able to enumerate all simplicial 4-polytopes with 10 vertices and neighborly simplicial d-polytopes with n vertices for the pairs (d, k)=$(4, 11)$, $(5, 10)$, $(6, 11)$ and $(7, 11)$, see [1]. We also decided for almost all enumerated polytopes, whether they can be realized with all vertices on the unit sphere. We will indicate how these results were obtained using optimization techniques and outline possible future applications.

[1] Moritz Firsching. Realizability and inscribability for some simplicial spheres and matroid polytopes. arXiv:1508.02531

Affine Symmetries of Orbit Polytopes

Erik Friese

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(joint work with Frieder Ladisch)

An orbit polytope is the convex hull of an orbit under a finite group $G \leq \mathrm{GL}(d, \mathbb{R})$. We develop a general theory of possible affine symmetry groups of orbit polytopes. For every group, we define an open and dense set of generic points such that the orbit polytopes of generic points have similar affine symmetry groups. We show how to compute the affine symmetries of generic orbit polytopes just from the character of G. We prove that any symmetry group of a generic point is equal to G if G is itself the affine symmetry group of some orbit polytope.

We use our theory to classify all finite groups which arise as affine symmetry groups of orbit polytopes. The only groups arising not in that way are elementary abelian groups of order 4, 8, and 16, abelian groups of exponent greater than 2, and generalized dicyclic groups. This answers a question of Babai who classified the orthogonal symmetry groups of orbit polytopes.

[1] Erik Friese, Frieder Ladisch. Affine Symmetries of Orbit Polytopes. arXiv:1411.0899

Volumes of flexible polyhedra in Lobachevsky spaces

Alexander Gaifullin

Steklov Mathematical Institute

A *flexible polyhedron* in the n-dimensional space is an $(n-1)$-dimensional closed polyhedral surface that can be deformed continuously so that every its face remains congruent to itself during the deformation, but the deformation is not induced by an ambient rotation of the space. Intuitively, one may think of a flexible polyhedron as of a polyhedral surface with faces made of some rigid material and with hinges at edges that allow dihedral angles to change continuously. However, this surface may be self-intersected. This definition can be used in all spaces of constant curvature, namely in the Euclidean spaces $\mathbb{E}^n$, in the Lobachevsky spaces Λ^n, and in the round spheres $\mathbb{S}^n$.

One of the most interesting problems concerning flexible polyhedra is the so-called bellows conjecture stated by Connelly in 1978 that asserts that the volume of any flexible polyhedron (in dimensions greater than or equal to 3) is constant during the flexion. This conjecture was proved in the Euclidean spaces of all dimensions (Sabitov [4] for $n = 3$, and the author [2] for $n \geq 4$). Flexible polyhedra of non-constant volumes were found in all open hemispheres $\mathbb{S}^n_+$ (Alexandrov [1] for $n = 3$, and the author [3] for $n \geq 4$), thus disproving the bellows conjecture in $\mathbb{S}^n$.

Theorem. *The bellows conjecture is true for bounded flexible polyhedra in odd-dimensional Lobachevsky spaces, i. e., the volume of any bounded flexible polyhedron in Λ^n, where n is odd and $n \geq 3$, is constant during the flexion.*

[1] V. Alexandrov, "An Example of a Flexible Polyhedron with Nonconstant Volume in the Spherical Space", *Beitr. Algebra Geom.*, Vol. 38, No. 1, 11–18 (1997).

[2] A. A. Gaifullin, "Generalization of Sabitov's Theorem to Polyhedra of Arbitrary Dimensions", *Discrete Comput. Geom.*, Vol. 52, No. 2, 195–220 (2014).

[3] A. A. Gaifullin, "Embedded flexible spherical cross-polytopes with nonconstant volumes", *Proc. Steklov Inst. Math.*, Vol. 288 (2015), 56–80.

[4] I. Kh. Sabitov, "Volume of a polyhedron as a function of its metric", *Fundam. Appl. Math.* Vol. 2, No. 4, 1235–1246 (1996) (in Russian).

Linear and semidefinite relaxations for spherical codes

Alexey Glazyrin

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(joint work with Oleg Musin)

An n-dimensional spherical code is a subset of points on the unit sphere in $\mathbb{R}^n$. A natural way to define spherical codes is through their Gram matrices. Matrix T corresponds to an n-dimensional spherical code if and only if it has 1's on the main diagonal, all entries are between -1 and 1, T is positive-semidefinite and its rank is no greater than n.

For various extremal problems on spherical codes, almost all these conditions on T are fairly easy to be taken into account since they are linear or semidefinite. Unfortunately, it is quite complicated to check the rank condition. Standard methods such as the Delsarte method relax the rank condition to linear or semidefinite conditions. The point of our work was to determine the gap between the exact description of codes and the relaxed ones. As the main result of this research, we showed that semidefinite relaxations can substitute the rank condition for $n = 2$ but not for larger n.

Asymptotic lower bound and parametric family of weighted spherical designs

Dmitry Gorbachev

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(joint work with Andriy Bondarenko)

We study the quadrature formulas

$$|S^d|^{-1} \int_{S^d} f(x)\,dx = \sum_{\nu=1}^{N} \lambda_\nu f(x_\nu),$$

where S^d is the Euclidean sphere, $f \in \mathbb{R}[x_1, \dots, x_{d+1}]$, $\deg f \le s$, $x_\nu \in S^d$ and $\lambda_\nu \ge 0$ are the nodes and weights respectively. The set $X = \{(x_\nu, \lambda_\nu)\}_{\nu=1}^{N}$ is called the (weighted) spherical s-design. Let

$$N_d(s) := \min_{X \text{ is } s\text{-design}} |X|.$$

For $\lambda_1 = \dots = \lambda_N$, Seymour and Zaslavsky [9] proved that $N_d(s) < +\infty$. Recently, Bondarenko, Radchenko, and Viazovska [1] obtained that

$$\limsup_{s \to +\infty} s^{-d} N_d(s) < +\infty$$

(the conjecture of Korevaar and Meyers [6]).

On the other hand, Delsarte, Goethals, and Seidel [4] proved that $c_d \gtrsim \log_2(e/2) \ge 0.4426$, where

$$c_d := d^{-1} \log_2 \left(d^d \liminf_{s \to +\infty} s^{-d} N_d(s) \right).$$

Later Yudin [10] obtained that $c_d \gtrsim 1$. *We show that*

$$c_d \gtrsim \log_2 \left(e(\overline{\Delta}_d)^{-1/d}/2 \right) \ge 1.0416,$$

where $\overline{\Delta}_d$ is the linear program bound for the density Δ_d of sphere packing [5], [3].

Also we present a parametric family X_{λ_1,λ_2} of minimal weighted 4-*designs consisting* 10 *points on the sphere* S^2 [2] and some close results. The distribution of the nodes of X_{λ_1,λ_2} is defined by polynomials from $\mathbb{Z}[t,\lambda_1^{-1},\lambda_2^{-1}]$. Our family includes two constructions from [7] ($X_{1/12,1/12}$) and [8] ($X_{1/9,1/9}$).

The first author were partially supported by RFBR (project N 13-01-00045), Ministry of education and science of Russian Federation (projects N 5414GZ), and Dmitry Zimin's Dynasty Foundation.

[1] A. Bondarenko, D. Radchenko, and M. Viazovska, *Optimal asymptotic bounds for spherical designs*, Ann. Math., **178** (2013), no. 2, 443–452.

[2] A. V. Bondarenko and D. V. Gorbachev, *Minimal weighted* 4-*designs on the sphere* S^2, Math. Notes, **91** (2012), no. 5–6, 738–741.

[3] H. Cohn and N. Elkies, *New upper bounds on sphere packings I*, Ann. Math. **157** (2003), no. 2, 689–714.

[4] P. Delsarte, J. M. Goethals, and J. J. Seidel, *Spherical codes and designs*, Geom. Dedicata, **6** (1977), 363–388.

[5] D. V. Gorbachev, *An extremal problem for entire functions of exponential spherical type, which is connected with the Levenshtein bound for the density of a packing of* $\mathbb{R}^n$ *by balls* (Russian), Izv. Tul. Gos. Univ. Ser. Mat. Mekh. Inform., **6** (2000), no. 1, 71–78.

[6] J. Korevaar, J. L. H. Meyers, *Spherical Faraday cage for the case of equal point charges and Chebyshev-type quadrature on the sphere*, Integral Transforms Spec. Funct., **1** (1993), no. 2, 105–117.

[7] A. S. Popov, *Cubature formulae for a sphere invariant under cyclic rotation groups*, Russ. J. Numer.Anal. Math. Modelling, **9** (1994), no. 6, 535–546.

[8] Sangwoo Heo and Yuan Xu, *Constructing cubature formulae for spheres and balls*, J. Comp. Appl. Math., **112**, no. 1–2 (1999), 95–119.

[9] P. D. Seymour and T. Zaslavsky, *Averaging sets: a generalization of mean values and spherical designs*, Adv. Math., **52** (1984), 213–240.

[10] V. A. Yudin, *Lower bounds for spherical designs*, Izv. Math., **61** (1997), no. 3, 673–683.

The Grünbaum–Hadwiger–Ramos hyperplane mass partition problem

Albert Haase

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(joint work with Pavle V. M. Blagojević, Florian Frick, Günter M. Ziegler)

How should a version of the ham-sandwich theorem for an arbitrary number of measures and hyperplanes be phrased? This question goes back to Grünbaum (1960) [3]. It lead to the *Grünbaum–Hadwiger–Ramos hyperplane mass partition problem*: For each $j \geq 1$ and $k \geq 1$, determine the smallest dimension $d = \Delta(j,k)$ such that for every collection of j masses on $\mathbb{R}^d$ there are k affine hyperplanes that cut each of the j masses into 2^k equal pieces. In this context masses are usually assumed to be probability Borel measures with connected support that vanish on hyperplanes.

Bounds for $\Delta(j,k)$ have been established by Avis (for $j=1$) [1] and Ramos [6] (lower bounds), and Mani-Levitska, Vrećica & Živaljević [5] (upper bounds):

$$\left\lceil \tfrac{2^k-1}{k} j \right\rceil \;\leq\; \Delta(j,k) \;\leq\; j + (2^{k-1}-1)2^{\lfloor \log_2 j \rfloor}.$$

Here $2^{\lfloor \log_2 j \rfloor}$ is j "rounded down to the next power of 2," so $\frac{1}{2}j < 2^{\lfloor \log_2 j \rfloor} \leq j$. However, few exact values of $\Delta(j,k)$ are known [2, 4, 5, 6].

And, more suprisingly, all known exact values are equal to the lower bound, which is obtained by a simple general position argument.

In this talk I explain how we were able to obtain new values for $\Delta(j,k)$ by employing equivariant relative obstruction theory to show that a certain "test map" without zeros cannot exist.

[1] David Avis. Non-partitionable point sets, Inform. Process. Letters 19 (1984), no. 3, 125-129.

[2] Pavle V. M. Blagojević, Florian Frick, Albert Haase, Günter M. Ziegler, Topology of the Grünbaum–Hadwiger–Ramos hyperplane mass partition problem, Preprint, 27 pages, February 2015. arXiv:1502.02975

[3] Branko Grünbaum, Partitions of mass-distributions and of convex bodies by hyperplanes. Pacific J. Math. 10 (1960), no. 4, 1257-1261.

[4] Hugo Hadwiger, Simultane Vierteilung zweier Körper, Arch. Math. 17 (1966), no. 3, 274-278.

[5] Peter Mani-Levitska, Siniš Vrećica, Rade T. Živaljević, Topology and combinatorics of partitions of masses by hyperplanes, Adv. Math. 207 (2006), no. 1, 266-296.

[6] Edgar A. Ramos, Equipartitions of mass distributions by hyperplanes, Discrete Comput. Geom. 15 (1996), no. 2, 147-167.

Orthogonal colorings of the sphere

Andreas Holmsen

KAIST, Daejeon and EPFL, Lausanne

(joint work with Seunghun Lee)

An orthogonal k-coloring of the two-dimensional unit sphere $\mathbb{S}^2$, is a partition of $\mathbb{S}^2$ into k parts such that no part contains a pair of orthogonal points, that is, a pair of points at spherical distance $\pi/2$ apart. It is a simple and well-known result that an orthogonal coloring of $\mathbb{S}^2$ requires at least four parts, and orthogonal 4-colorings can easily be constructed from a regular octahedron centered at the origin. An intriguing and natural question is whether or not every orthogonal 4-coloring of $\mathbb{S}^2$ is such an octahedral coloring.

In this talk I will give several characterizations of orthogonal 4-colorings of $\mathbb{S}^2$ which are octahedral. For instance, if every color class has a non-empty interior, then the coloring is octahedral.

In constrast to these results I will give an example of an orthogonal 9-coloring of $\mathbb{S}^2$ where each color class is dense in $\mathbb{S}^2$.

[1] Andreas Holmsen and Seunghun Lee. Orthogonal colorings of the sphere. arXiv:1505.02514

Limits of order types

Alfredo Hubard

Institute National de Recherche en Informatique et en Automatique, Sophia-Antipolis

(joint work with Xavier Goaoc, Rémi de Joannis de Verclos Jean-Sébastien Séréni, and Jan Volec)

Order types are invariants of finite point sets that appear in a number of contexts in combinatorial and computational geometry. In this talk I will consider order types from the point of view of limits of combinatorial objects.

Using the flag algebra framework framework we obtain some new concrete results to old combinatorial problems. Guided by analogies with the theory of graphons, we draw a number of connections with measure theory.

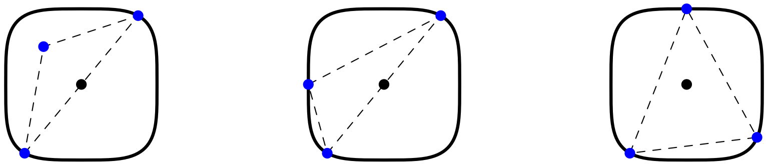
The Center Problem in Strictly Convex Planes

Thomas Jahn

Technische Universität Chemnitz

Approximating a given finite set of points by a single point (called the *center*) in a minimax fashion is an important task in operation research and statistical analysis. The problem can be solved by finding centers of triples of given points. This inherent discrete geometric structure is utilized by an algorithm designed for the Euclidean plane in the early 1970s. It turns out that this algorithm works well in any two-dimensional vector space equipped with a strictly convex norm. The talk is based on the paper

[1] Thomas Jahn. Geometric Algorithms For Minimal Enclosing Discs In Strictly Convex Normed Planes. Contributions to Discrete Mathematics, to appear.



Balanced generalized lower bound inequality for simplicial polytopes

Martina Juhnke-Kubitzke

Universität Osnabrück

(joint work with Satoshi Murai)

A remarkable and important property of face numbers of simplicial polytopes is the generalized lower bound inequality, which says that the h-vector of any simplicial polytope is unimodal. Recently, for balanced simplicial d-polytopes, that is simplicial d-polytopes whose underlying graph is d-colorable, Klee and Novik proposed a balanced analogue of this inequality, that is stronger than just unimodality.

In order to prove this conjecture, we will show a Lefschetz property for rank-selected subcomplexes of balanced simplicial polytopes and thereby obtain new inequalities for their h-vectors.

The relevant publications on this topic are listed below.

[1] M. Juhnke-Kubitzke and S. Murai, Balanced generalized lower bound inequality for simplicial polytopes, arXiv:1503.06430

[2] S. Klee and I. Novik, Lower Bound Theorems and a Generalized Lower Bound Conjecture for balanced simplicial complexes, arXiv:1409:5094.

[MW] P. McMullen and D.W. Walkup, A generalized lower-bound conjecture for simplicial polytopes, *Mathematika* **18** (1971), 264–273.

[3] S. Murai and E. Nevo, On the generalized lower bound conjecture for polytopes and spheres, *Acta Math.* **210** (2013), 185–202.

[St2] R.P. Stanley, The number of faces of simplicial convex polytopes, *Adv. Math.* **35** (1980), 236–238.

Recent results on local h-vectors

Lukas Katthän

Universität Osnabrück

(joint work with Martina Juhnke-Kubitzke and Richard Sieg)

Let Δ be a (finite) simplicial complex. The f-vector of Δ counts the number of faces of each dimension in Δ. Often it is more convenient to consider the h-vector instead, which is obtained from the f-vector by a certain linear transformation. In order to understand the change of the h-vector under a subdivision of Δ, the local h-vector was introduced by Kalai and Stanley. This local h-vector is an invariant associated to a subdivision of a single simplex.

In this talk, I will present some recent results about the set of possible local h-vectors for particular classes of subdivisions. This is joint work with Martina Juhnke-Kubitzke and Richard Sieg.

Translative covering of the space with slabs

Andrey Kupavskii

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(joint work with János Pach)

The set of points S lying between two parallel hyperplanes in $\mathbb{R}^d$ at distance w from each other is called a *slab* of *width* w. We say that a sequence of slabs $S_1, S_2, \dots$ permits a *translative covering* of a subset $B \subseteq \mathbb{R}^d$ if there are suitable translates S'_i of S_i $(i = 1, 2, \dots)$ such that $B \subseteq \cup_{i=1}^{\infty} S'_i$.

It was shown by Makai and Pach [2] and, independently, by Erdős and Straus (unpublished, see [1]) that any sequence of slabs whose total weight is divergent permits a translative covering of the whole plane. Actually, they showed that there is a constant $c > 0$ such that any system of slabs in the plane with total width at least c permits a translative covering of a disk of diameter 1. This result may be seen as a dual to the the famous Tarski's result [3], which states that the total width of any system of slabs that cover a disk of unit diameter is at least 1.

As for the higher dimensions, Makai and Pach conjectured that

Conjecture 1. (Makai-Pach) Let d be a positive integer. A sequence of slabs in $\mathbb{R}^d$ with widths $w_1, w_2, \dots$ permits a translative covering of $\mathbb{R}^d$ if and only if $\sum_{i=1}^{\infty} w_i = \infty$.

Let $w_1 \geq w_2 \geq \dots$ be a monotone decreasing sequence of positive numbers. Improving some earlier results of Groemer, we prove the following theorem

Theorem 1. *Let d be a positive integer, and let $w_1 \geq w_2 \geq \dots$ be a monotone decreasing infinite sequence of positive numbers such that*

$$\limsup_{n\to\infty} \frac{w_1 + w_2 + \dots + w_n}{\log(1/w_n)} > 0.$$

Then any sequence of slabs S_i of width w_i $(i = 1, 2, \dots)$ permits a translative covering of $\mathbb{R}^d$.

Let $\mathcal{P}_d$ denote the family of all polynomials of degree at most d in one variable x, with real coefficients. A sequence of positive numbers $x_1 \le x_2 \le \ldots$ is called *$\mathcal{P}_d$-controlling* if there exist $y_1, y_2, \ldots \in \mathbb{R}$ such that for every polynomial $p \in \mathcal{P}_d$ there exists an index i with $|p(x_i) - y_i| \le 1$. In fact, the notion of controlling sequences for different classes of functions is closely related to the first topic of this abstract, translative coverings of the space by slabs. We settle a problem from Makai and Pach's paper in the following theorem.

Theorem 2. *Let d be a positive integer and $x_1 \le x_2 \le \ldots$ be a monotone increasing infinite sequence of positive numbers. The sequence $x_1, x_2, \ldots$ is* $\mathcal{P}_d$-controlling *if and only if*

$$\lim_{n\to\infty} (x_1^{-d} + x_2^{-d} + \ldots + x_n^{-d}) = \infty.$$

[1] H. Groemer, *On coverings of convex sets by translates of slabs,* Proc. Amer. Math. Soc. **82** (1981), no. 2, 261–266.

[2] E. Makai Jr. and J. Pach, *Controlling function classes and covering Euclidean space*, Stud. Scient. Math. Hungarica **18** (1983), 435–459.

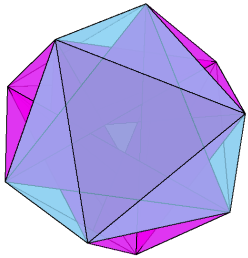
[3] A. Tarski, Uwagi o stopnii równowaznosci wielokatów, *Parametr* **2** (1932), 310–314.

Vertex-Transitive Polyhedra

Undine Leopold

Technische Universität Chemnitz

The classification of vertex-transitive polyhedra of genus $g \ge 2$ in Euclidean 3-space, started by Grünbaum and Shephard in 1984 [1], has proven to be a difficult problem despite its rigid setting. Here, polyhedra are face-to-face tessellations of closed, connected, orientable, embedded surfaces by simple, plane polygons. Seven examples are known, and it is also known that the symmetry groups must be among the rotation groups of the Platonic solids [2]. The investigation of tetrahedral rotation symmetry has been completed [3, 4], with no new examples besides the known polyhedron of genus 3.



However, many more candidate maps can be enumerated for the octahedral and icosahedral case. In this talk, I will highlight the connections between symmetry, geometry, and topology for candidate maps and polyhedra, and present an overview of recent progress.

[1] B. Grünbaum and G. C. Shephard. Polyhedra with Transitivity Properties. *C. R. Math. Rep. Acad. Sci. Canada*, 6(2):61–66, 1984.

[2] Gábor Gévay, Egon Schulte, and Jörg M. Wills. The regular Grünbaum polyhedron of genus 5. *Adv. Geom.*, 14(3):465–482, 2014.

[3] Undine Leopold. *Vertex-Transitive Polyhedra in Three-Space*. ProQuest LLC, Ann Arbor, MI, 2014. Thesis (Ph.D.)–Northeastern University.

[4] Undine Leopold. Vertex-Transitive Polyhedra of Higher Genus, I. arXiv:1502.07497

Half-space depth of a line

Alexander Magazinov

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(joint work with Attila Pór)

Let α be a k-flat and μ a probabilistic measure in $\mathbb{R}^d$ $(0 \le k < d)$. Define the *depth* of α as follows:

$$\operatorname{depth}(\alpha) = \inf\{\mu(H) : H \text{ is a closed half-space, } \alpha \subset \partial H\}.$$

To distinguish with other notions of depth, the above defined depth is sometimes called *half-space depth* or *Tukey depth*.

Bukh, Matoušek and Nivasch [1] proposed the following conjecture:

Conjecture 2. Let a pair of integers (d, k) with $0 \le k < d$ be given. Then for every probabilistic measure μ in $\mathbb{R}^d$ there exists a k-flat α in $\mathbb{R}^d$ (a *centerflat*) such that

$$\operatorname{depth}(\alpha) \ge \frac{k+1}{k+d+1}.$$

The conjecture is true for $k = 0$ (Rado's centerpoint theorem, 1946, see [4]), $k = d-1$ (trivial), and $k = d-2$ (due to Bukh, Matoušek and Nivasch [1]).

A result by Klartag [3] implies that, if $d-k$ is fixed, then for every $\varepsilon > 0$, with d sufficiently large depending on ε, and for every probabilistic measure μ in $\mathbb{R}^d$ there exists a k-flat α in $\mathbb{R}^d$ such that

$$\operatorname{depth}(\alpha) > \frac{1}{2} - \varepsilon.$$

For $k = 0$ and $k = d-1$ the constant $\frac{k+1}{k+d+1}$ cannot be increased. Buch and Nivash [2] have proved that for $k = 1$ the constant $\frac{k+1}{k+d+1} = \frac{2}{d+2}$ also cannot be increased.

Due to Rado's centerpoint theorem, for every d and k and μ, one can find a k-flat α (in fact, in any direction) such that

$$\mathrm{depth}(\alpha) \geq \frac{1}{d-k+1}$$

(the trivial bound).

In this talk we announce the result that for $k \geq 1$ the trivial bound is not optimal. Certainly, it is enough to consider just the case $k = 1$. Namely, we have the following result:

Theorem 3. *There exists a function $c(d) > 0$ such that for every probabilistic measure μ in $\mathbb{R}^d$ there is a (1-dimensional) line α with*

$$\mathrm{depth}(\alpha) \geq \frac{1}{d} + c(d).$$

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[3] Bo'az Klartag. On nearly radial marginals of high-dimensional probability measures. *Journal of the European Mathematical Society* 12 (2010), 723–754.

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About lower bound for the number of facets of a k-neighborly polytope

Alexander Maksimenko

P.G. Demidov Yaroslavl State University

Let P be a d-polytope with $f_0(P)$ vertices and let $f_i(P)$ be the number of i-faces of P, $1 \leq i \leq d-1$. The problem of estimating $f_i(P)$ (where P belongs to some class of polytopes) in terms of $f_0(P)$ is well known. The solutions for the class of simplicial polytopes are known as the upper bound and the lower bound theorems (see [1] for details). In 1990, G. Blind and R. Blind [2] solved the upper bound problem for the class of polytopes without a triangle 2-face. We raise the question for the class of 2-neighborly polytopes.

A d-polytope P is called *k-neighborly* polytope if each subset of k vertices forms the vertex set of some face of P. If, in addition, $k = \lfloor d/2 \rfloor$, then P is called neighborly polytope. In particular, for $d \geq 4$ every neighborly d-polytope is 2-neighborly. The family of neighborly polytopes are investigated very intensively (see, e.g., [1]). It is seems that k-neighborly polytopes are very common among convex polytopes [3, 4]. Moreover, they appear as faces (with superpolynomial number of vertices) in many known combinatorial polytopes, associated with NP-complete problems [5, 6].

As a reference point we pose the following conjecture.

Conjecture 3. The number of facets $f_{d-1}(P)$ of a k-neighborly polytope P cann't be less than the number of its vertices $f_0(P)$ for $k \geq 2$.

Proposition 4. *The conjecture is true for $d \leq 2k+2$.*

Theorem 5. *$f_{d-1}(P) \geq d + k^2 + 1$ for a k-neighborly d-polytope P with $f_0(P) \geq d + 2$.*

With the help of Gale diagrams we have found the tight lower bound for the case $f_0(P) = d + 3$.

Theorem 6. *If P is a k-neighborly d-polytope with $f_0(P) = d+3$, then*

$$f_{d-1}(P) - f_0(P) \geq \begin{cases} 2(k^2-1) & \text{for } k \geq 4, \\ (k+2)(k^2+k-3)/3 & \text{for } k \in \{2,3\}. \end{cases}$$

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[2] G. Blind and R. Blind. Convex polytopes without triangular faces. Isr. J. Math. 1990, V. 71, P. 129–134.

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Discrete Geometry in Minkowski Spaces

Horst Martini

TU Chemnitz (Germany)

In recent decades, many papers appeared in which typical problems of Discrete Geometry are investigated, but referring to finite dimensional real Banach spaces (i.e., to Minkowski Geometry) or, even more general, to spaces with so-called asymmetric norms (gauges).

In many cases the extension of basic geometric notions, needed for posing these problems in non-Euclidean Banach spaces, is already interesting enough. Examples of such notions and problems are: circumballs and -centers of convex sets (e.g., studying Chebyshev sets), corresponding inballs and -centers, packings and coverings (for instance, Lebesgue's universal covering problem), problems from Location Science (like minsum hyperplanes and minsum hyperspheres), properties of curves and surfaces in the spirit of Discrete Differential Geometry, reduced and complete sets (e.g., for polyhedral norms), applications of notions from Combinatorial Geometry (such as Helly dimension), and generalized theorems from incidence geometry (e.g., the theorems of Clifford and Miquel).

In this talk, an overview to several such problems and related needed notions is given.

Solving Mordell equations via the Shimura–Taniyama conjecture

Benjamin Matschke

Max Planck Institute for Mathematics, Bonn

(joint work with Rafael von Känel)

In this talk certain discrete geometry (or geometry of numbers) aspects of the project [2] will be presented. One of two main goals of this project is to practically solve the classical Mordell equation

$$y^2 = x^3 + a \tag{1}$$

over the integers (and more generally over the S-integers) for any given integer $a \neq 0$. Using the Shimura–Taniyama conjecture and a method of Faltings [1] (Arakelov, Paršin, Szpiro) we obtain new height bounds for the solutions (x, y) of (1), improving on previous bounds that were derived from the theory of linear forms in logarithms.

Using these height bounds, only finitely many candidates (x, y) for equation (1) remain, however their number is still huge. In order to reduce the search space further, practical sieves have been developed by de Weger [3], Zagier, Stroeker–Tzanakis, Gebel–Pethő–Zimmer, and others. In [2] the so-called elliptic logarithm sieve is constructed, which improves in several ways the previous sieves and thus yields a faster algorithm.

One of these improvement motivates the following problem in discrete geometry, related to non-convex polytopes.

Problem. Let $A := \mathbb{R}^n_{\geq 0}$. Given $k \geq n$, how does one choose $x_1, \ldots, x_k \in A$ with $||x_i||_1 = 1$ such that $\sup\{||a||_1 \,:\, a \in A \setminus \bigcup_i (x_i + A)\}$ is minimal?

As the application of this problem is an algorithm, we do not need the exact answer to this problem; an approximate solution is good enough.

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Variations of the nerve theorem and Meshulam–Sperner type results

Luis Montejano

National University of Mexico at Querétaro

Let K be a simplicial complex. Suppose the vertices of K are painted with $I = \{1, \ldots, m\}$ colors. We study the existence of a rainbow simplex of K under the hypothesis that certain homology groups of certain subcomplexes of K are zero. By using this ideas, we are able to prove several geometric Hall-type results.

Proof of a conjecure of Bárány, Katchalski, and Pach

Márton Naszódi

Eötvös University, Budapest and EPFL, Lausanne

Bárány, Katchalski, and Pach proved the following quantitative form of Helly's theorem: If the intersection of a family of convex sets in $\mathbb{R}^d$ is of volume one, then the intersection of some subfamily of at most $2d$ members is of volume at most some constant $v(d)$. They gave the bound $v(d) \leq d^{2d^2}$, and conjectured that $v(d) \leq d^{cd}$. We confirm it.

[1] I. Bárány, M. Katchalski, and J. Pach, *Quantitative Helly-type theorems*, Proc. Amer. Math. Soc. **86** (1982), no. 1, 109–114.

[2] I. Bárány, M. Katchalski, and J. Pach, *Helly's theorem with volumes*, Amer. Math. Monthly **91** (1984), no. 6, 362–365.

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Volume and lattice points counting for the cyclopermutohedron

Ilia Nekrasov

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(joint work with Gaiane Panina)

The standard permutohedron Π_n is defined as the convex hull of all points in $\mathbb{R}^n$ that are obtained by permuting the coordinates of the point $(1, 2, \ldots, n)$. The face lattice of the permutohedron realizes the combinatorics of linearly ordered partitions of the set $[n] = \{1, \ldots, n\}$.

Similarly, *the cyclopermutohedron* $\mathcal{CP}_{n+1}$ [8] is a virtual polytope (see [9]) which realizes the combinatorics of cyclically ordered partitions of the set $[n]$: k-faces are labeled by (all possible) cyclically ordered partitions of the set $[n+1] = \{1, \ldots, n, n+1\}$ into exactly $(n+1-k)$ non-empty parts, where $(n+1-k) > 2$. The incidence relation in $\mathcal{CP}_{n+1}$ (like the "permutohedron case") corresponds to the refinement of partitions: a cell F contains a cell F' whenever the label of F' refines the label of F.

It is known that the volume of the *standard permutohedron* can be expressed in terms of the number of trees with n labeled vertices. The number of integer points of the standard permutohedron equals the number of forests on n labeled vertices.

The purpose of this talk is to show that the volume of the cyclopermutohedron also equals some weighted number of forests, which eventually reduces to zero for all n. We also derive a combinatorial formula for the number of integer points in the cyclopermutohedron (see [5]).

Acknowledgment. The present research is supported by Russian Foundation for Basic Research project 15-01-02021, by the Chebyshev Laboratory under RF Government grant 11.G34.31.0026, and JSC "Gazprom Neft".

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[3] A. Khovanskii. Newton polyhedra and toroidal varieties. Functional Analysis and Its Applications, 11(4):289–296, 1977.

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[5] I. Nekrasov and G. Panina. Volume and lattice points counting for the cyclopermutohedron. arXiv:1505.00352

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Accidental Meetings

János Pach

Ecole Polytechnique Fédérale de Lausanne and Alfred Renyi Institute of Mathematics, Budapest

(joint work with Natan Rubin and Gábor Tardos)

If two closed Jordan curves in the plane have precisely one point in common, then it is called a touching point. All other intersection points are called crossing points. We establish a Crossing Lemma for closed curves: In any family of n pairwise intersecting simple closed curves in the plane, no three of which pass through the same point, the number of crossing points exceeds the number of touching points by a factor that tends to infinity as n gets larger.

As a corollary, we prove the following long-standing conjecture of Richter and Thomassen: The total number of intersection points between any n pairwise intersecting simple closed curves in the plane, no three of which pass through the same point, is at least $(1 - o(1))n^2$.

Cyclopermutohedron

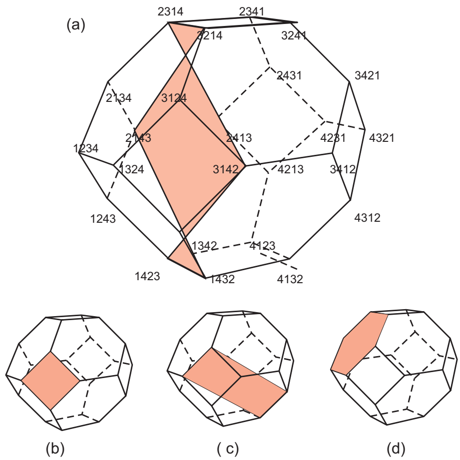
Gaiane Panina

Saint-Petersburg Institute for Informatics and Automation RAS, Saint Petersburg State University

It is known that the k-faces of the permutohedron Π_n can be labeled by (all possible) linearly ordered partitions of the set $[n] = \{1, ..., n\}$ into $(n-k)$ non-empty parts. The incidence relation corresponds to the refinement: a face F contains a face F' whenever the label of F' refines the label of F.

In the talk we consider the cell complex defined in analogous way, replacing linear ordering by cyclic ordering. Namely, the k-cells of the complex are labeled by (all possible) cyclically ordered partitions of the set $[n+1]$ into $(n+1-k)$ non-empty parts, where $(n+1-k)>2$. The incidence relation in the complex again corresponds to the refinement.

The complex cannot be represented by a convex polytope, since it is not a combinatorial sphere (not even a combinatorial manifold). However, it can be represented by some *virtual polytope* (that is, Minkowski difference of two convex polytopes) which we call *cyclopermutohedron*. It is defined explicitly, as a weighted Minkowski sum of line segments. Informally, the cyclopermutohedron can be viewed as "permutohedron with diagonals", see the figure. One of the motivations is that the cyclopermutohedron is a "universal" polytope for moduli spaces of polygonal linkages.



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Multilevel polynomial partitions

Zuzana Patáková

Charles University in Prague

(joint work with Jiří Matoušek)

The talk is devoted to the polynomial partitioning method of Guth and Katz, which partitions a given n-point set $P \subset \mathbb{R}^d$ using the zero set $Z(f)$ of a suitable d-variate polynomial f. Applications of this result are often complicated by the problem, what should be done with the points of P lying within $Z(f)$? A natural approach is to partition these points with another polynomial and continue further in a similar manner.

As a main result, we provide a polynomial partitioning method with up to d polynomials in dimension d, which allows for a complete decomposition of the given point set.

In more detail: given an n-point set $P \subset \mathbb{R}^d$ and a parameter $r > 1$, we say that a nonzero polynomial $f \in \mathbb{R}[x_1, \ldots, x_d]$ is a $\frac{1}{r}$-*partitioning polynomial* for P if none of the connected components of $\mathbb{R}^d \setminus Z(f)$ contains more than n/r points of P.

As mentioned before, it is crucial to deal with the situation when the point set $P \subset \mathbb{R}^d$ lies within a zero set: We show that given $r > 1$, a k-dimensional complex variety V whose all irreducible components have dimension k as well, and a finite point set $P \subset V \cap \mathbb{R}^d$, there exists a $\frac{1}{r}$-partitioning polynomial for P of degree at most $O(r^{1/k})$ that does not vanish on any of the irreducible components of V.

Assuming, moreover, that V is irreducible of degree Δ and $r \geq \Delta^{k+1}$, we can show that there exists a $\frac{1}{r}$-partitioning polynomial for P of degree at most $O((r/\Delta)^{1/k})$. This confirms a conjecture of Matoušek and the speaker mentioned in [2, Conj. 3.2].

The relevant publications on this topic are listed below.

[1] Larry Guth, Nets Katz: On the Erdős distinct distances problem in the plane. *Ann. Math.*, 181: 155–190, 2015.

[2] Jiří Matoušek, Zuzana Patáková: Multilevel polynomial partitions and simplified range searching. *Dis. Comp. Geom.*, 54(1):22–41, 2015.

[3] Zuzana Patáková: Problems in discrete geometry. Doctoral thesis.

INVARIANT ZONOIDS AND L_1 SPECTRAL RADIUS OF MATRICES

Vladimir Protasov

Moscow State University

Every irreducible set of linear operators $\{A_1, \ldots, A_m\}$ in $\mathbb{R}^d$ possesses an invariant zonoid $G \subset \mathbb{R}^d$ (the Minkowski sum of a countable set of segments) such that G is homothetic to the Minkowski sum of images $A_i G$, $i = 1, \ldots, m$. The coefficient of homothety is equal to the so-called L_1 spectral radius of those operators [2] This concept originated in 1995 with Wang [1] found numerous applications in functional analysis, approximation theory, probability, etc. We analyze the invariant zonoid G to compute or approximate the L_1-spectral radius. The existence of efficient methods of approximation follows from the results of Bourgain, Lindenstrauss, and Milman [4]. We consider applications to the problem of characterising matrix sets sharing a common invariant cone [3] and to the study of distributions of power random series with integer coefficients.

[1] Y.Wang. Two-scale dilation equations and the mean spectral radius. Random Comput. Dynamic, 4(1), 49–72.

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[3] V.Yu.Protasov. When do several linear operators share an invariant cone? Linear Alg. Appl., 433 (2010), 781–789. 2339-2357.

[4] J.Bourgain, J.Lindenstrauss, and V.Milman. Approximation of zonoids by zonotopes. Acta Mathematica, 162 (1989), no 1, 73–141.

Around Turán's theorem for some distance graphs

Philipp Pushnyakov

Moscow Institute of Physics and Technology, Faculty of Innovations and High Technology

In the talk, we consider a sequence of distance graphs

$$G(n) = (V(n), E(n)) :$$

$$V(n) = \{\mathbf{x} = (x_1, \ldots, x_n) : \ x_i \in \{0, 1\}, \ x_1 + \cdots + x_n = 3\},$$

$$E(n) = \{\{\mathbf{x}, \mathbf{y}\} : |\mathbf{x} - \mathbf{y}| = 2\},$$

where by $|\mathbf{x} - \mathbf{y}|$ we denote the Euclidean distance between vectors $\mathbf{x}, \mathbf{y}$. This sequence is deeply motivated by the Nelson–Hadwiger problem on coloring metric spaces.

We define $r(W)$ as the number of edges of a graph $G(n)$ on a subset of vertices $W \subset V(n)$. Also we put

$$r(l(n)) = \min_{|W| = l(n), \, W \subset V(n)} r(W).$$

In the talk, we will exhibit an almost exhaustive study of the quantity $r(l(n))$.

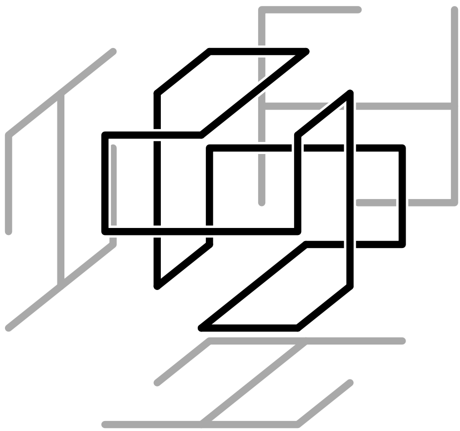
Shadows of a circle

Edgardo Roldán Pensado

Universidad Nacional Autónoma de México

(joint work with Michael G. Dobbins, Heuna Kim, and Luis Montejano)

Given an embedding of a topological space A in some Euclidean space of higher dimension, what does the topology of its shadows tell us about the topology of A? This is a very hard and general question. When A is a closed curve we show that it cannot have three linearly independent projections that are paths. However, it is possible to embed it so that its projections are trees as in the figure.



This curve is on the cover of [1] which also includes some of its history. The proof of our result is topological but uses nice and simple ideas.

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A general equilibrium approach to the multidimensional Tiebout hypothesis

Alexei Savvateev

Moscow Institute of Physics and Technology

(joint work with K. Sorokin, S. Weber)

Consider the following problem called 'multidimentional group formation'. We are given a probabilistic distribution over a finite-dimentional convex compactum, $X \subset \mathbf{R}^d$. This distribution is assumed to admit continuous density $f : X \to \mathbf{R}$, which is in addition bounded away from zero: $\exists \delta : \quad \forall x \in X \;\; f(x) \geq \delta$.

We look for a stable, 'migration-proof' partition of X into a prescribed number n of nonempty measurable compacta, $X = S_1 \cup \cdots \cup S_n$, almost mutually exclusive. Stability is meant in a game-theoretic sense, when the "centers" of those groups are given via $m_1, \ldots, m_n$ and each point $x \in X$ is interpreted as a citizen choosing between those n jurisdictions.

In searching for the stable configuration, citizen x compares cost functions in all the jurisdictions, cost functions which split into "monetary" and "transportation" parts:

$$\frac{\text{const}}{\int_{S_i} f(y)dy} + \ell(x, m_i),$$

where $\ell(\cdot, \cdot)$ is a given metric over X, and picks up one of jurisdictions which minimize costs. Above, a spectacular feature is the inverse proportionality between the monetary part of the cost and the measure of a jurisdiction, its "population". By using a technique borrowed from general-equilibrium theory, we prove the existence result.

Functional affine-isoperimetry and an inverse logarithmic Sobolev inequality

Carsten Schütt

Universität Kiel

We give a functional version of the affine isoperimetric inequality for log-concave functions which may be interpreted as an inverse form of a logarithmic Sobolev inequality inequality for entropy. A linearization of this inequality gives an inverse inequality to the Poincaré inequality for the Gaussian measure.

Distance graphs in the plane

Lev Shabanov

Higher School of Economics, Faculty of Mathematics

By a *(unit) distance graph* in the Euclidean plane $\mathbb{R}^2$ we mean a graph $G = (V, E)$ with $V \subset \mathbb{R}^2$ and $E = \{\{\mathbf{x}, \mathbf{y}\} : |\mathbf{x} - \mathbf{y}| = 1\}$. The classical problem by Nelson and Hadwiger is in finding the maximum chromatic number $\chi(G)$ of a distance graph G, and it is only known that this maximum is between 4 and 7. For the lower bound, the inequality $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$ is used, where $\alpha(G)$ is the independence number of G. In our talk, we discuss new general relations between the independence number, the number of vertices and the number of edges of a distance graph in the plane. In particular, we show that given the number of vertices and the independence number, the number of edges is much larger than the Turán bound. For example, if there exist graphs G_n with $4n$ vertices and $\alpha(G_n) \leq n$, then the number of edges is at least $\frac{26}{3}n$ instead of Turán's $6n$.

Local formulas for the Chern classes of triangulated S^1-bundles.

Georgy Sharygin

Moscow State University

(joint work with Nikolay Mnëv)

Let $\pi : L \to K$ be a map of simplicial complexes, whose geometric realization is homeomorphic to a locally trivial fiber bundle. There is a long-standing problem to express the characteristic classes of this bundle in terms of the combinatorics of this map. In my talk I will describe the solution of this problem in the simplest case (when the fiber is equal to the circle S^1) and give few insights into the possible solutions of the general case.

A classification of link maps of graphs to $\mathbb{R}^3$ and polyhedra to $\mathbb{R}^m$

Arkadiy Skopenkov

Moscow Institute of Physics and Technology and Independent University of Moscow

Let P and Q be polyhedra, i.e. bodies of simplicial complexes. A *link map* is a map $f : P \sqcup Q \to \mathbb{R}^m$ such that $f(P) \cap f(Q) = \emptyset$. A *link homotopy* is a homotopy through link maps.

For connected graphs P and Q, linking coefficients define a 1–1 correspondence between the set of link homotopy classes of link maps $f : P \sqcup Q \to \mathbb{R}^m$ and $\mathbb{Z}^{(\chi(P)+1)(\chi(Q)+1)}$, where χ is the Euler characteristic.

Although this result is simple (and, for this reason, may be folklore), the proof involves 3-dimensional visualization of the celebrated 4-dimensional Casson's finger moves.

Main Theorem (a particular case). [1] *If P and Q are closed orientable 2- and 3-manifolds then linking coefficients define a 2–1 map between the set of link homotopy classes of link maps $f : P \sqcup Q \to \mathbb{R}^5$ and $H_1(P) \oplus H_2(Q)$, where H_* is the homology group with $\mathbb{Z}$-coefficients.*

The proof involves higher-dimensional generalizations of the Whitney trick and Casson's finger moves.

I shall discuss recent generalizations joint with S. Avvakumov, I. Mabillard, U. Wagner, and possibly others. These generalizations concern maps $S^3 \sqcup S^3 \sqcup S^3 \to \mathbb{R}^5$ without triple intersections and Tverberg maps (or almost embeddings) of 2-dimensional simplicial complexes to $\mathbb{R}^4$.

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Quasiconvex hull of three points on the plane

Alexey Stepanov

V. Vernadsky Crimea Federal University

(joint work with Popova Elena)

It is known quite a lot of problems which solution is connected with the construction of the convex hull of a set of points on the plane. Therefore a task of finding optimal algorithms depending on problem specification is natural and urgent. However, in all certain papers on this topic all objects connected with convex hulls are considered as points, i.e. their size is neglected. And we plan to take into account the size of the objects.

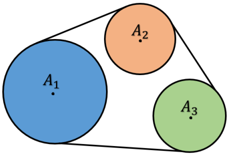

Fig. 1

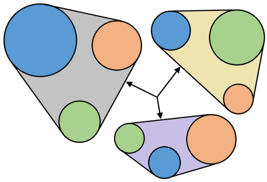
Fig. 2

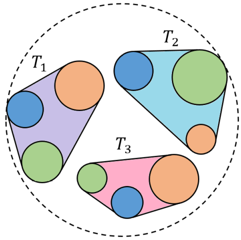

Fig. 3

We call the figure on Fig. 1 *quasiconvex hull* of the points A_1, A_2, A_3 with radii r_1, r_2, r_3 (see Fig. 1). We suggest analogues of Jarvis' and Graham's algorithms for constructing a quasiconvex hull of three points with given radii. Also we prove a solvability of analogs for Fermat-Torricelli-Steiner problem (see Fig. 2) and for minimal covering ball problem (see Fig. 3) for several quasiconvex hulls.

Financial support. The research of the first author was partially supported by the grant of the President of the Russian Federation, the code is MK-2915.2015.1 (the author took part in this research as a co-executor).

Some analogs of fair division problem

Fedor Stonyakin

V. Vernadsky Crimea Federal University

This talk is devoted to some analogs of the well-known fair division problem. The usual method to such problems is to consider subjects that use additive non-atomic and probabilistic measures for estimating the parts of divisible objects. Such studies are usually based on the A. A. Lyapunov Convexity Theorem in finite-dimensional spaces. We consider two types of problems.

Firstly, let's suppose that there are small enough sets with zero estimation. However, a union of such negligible sets may have nonzero estimation. In this case estimation function is not additive and non-atomic. To simulate this effect we propose two non-additive analogs of measure, namely *quasi-measure* and *ε-quasi-measure*. Corresponding analogs of the fair division problem are considered.

Secondly, we consider analog of the fair division problem for infinite number of measures. Namely, we have an object and σ-algebra of its subsets. Suppose there is a countable number of different criteria for estimating the divisible parts of the original object, but the object itself is identical in terms of these criteria. The question about "smoothing" of estimates for some set from the point of view of infinite criteria is considered. To consider this problem we use the special system of anticompact sets in Banach spaces introduced by us.

On the MacPhersonian

Ricardo Strausz

Universidad Nacional Autonoma de Mexico

In 2003 Daniel Biss published, in the Annals of Mathematics [1], what he thought was a solution of a long standing problem culminating a discovery by Gelfand and MacPherson [3]. Six years later he was encouraged to publish an "erratum" of his proof [2], observed by Nikolai Mnëv; up to now, the homotopy type of the so-called MacPhersonian had remained a mistery...

The aim of this lecture is to convince the attendee of the fact that, using a completely different approach to those used before, we can prove that the (acyclic) MacPhersonian has the homotopy type of the (affine) Grassmannian.

[1] Daniel K. Biss. The homotopy type of the matroid Grassmannian. *Annals of Mathematics* (2) 158, No. 3 (2003), 929–952.

[2] Daniel K. Biss. Erratum to "The homotopy type of the matroid Grassmannian". *Annals of Mathematics* 170 (2009), 493–493.

[3] I. M. Gelfand, Robert D. MacPherson. A combinatorial formula for the Pontrjagin classes. *Bull. Am. Math. Soc.* New Ser. 26, No. 2 (1992), 304–309.

Arrangements of homothets of a convex body

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(joint work with Márton Naszódi, János Pach)

In 1994 Füredi and Loeb [1] asked the following question:

> *Is it true that for any centrally symmetric body K of dimension d, $d \geq d_0$, the number of pairwise intersecting homothetic copies of K which do not contain each other's centers is at most 2^d?*

A construction of Talata [2] implies that the answer to this question is no. We find an upper bound that is asymptotically close to best possible. We also discuss the non-symmetric case and some related questions.

[1] Zoltán Füredi and Peter A. Loeb, *On the best constant for the Besicovitch covering theorem*, Proc. Amer. Math. Soc. **121** (1994), 1063–1073.

[2] István Talata, *On Hadwiger numbers of direct products of convex bodies*, Combinatorial and computational geometry, 517–528, Math. Sci. Res. Inst. Publ., **52**, Cambridge Univ. Press, Cambridge, 2005.

On computational complexity of length embeddability of graphs

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A graph G is embeddable in $\mathbb{R}^d$ if vertices of G can be assigned with points of $\mathbb{R}^d$ in such a way that all pairs of adjacent vertices are at the distance 1. We show that verifying embeddability of a given graph in $\mathbb{R}^d$ is NP-hard in the case $d > 2$ for all reasonable notions of embeddability. The same result was published in [1]. However, it relied essentially on an erroneous claim of Lovász (see [2], [3]), and we present a completely different construction.

[1] B. Horvat, J. Kratochvil, T. Pisanski. On the computational complexity of degenerate unit distance representations of graphs. *Combinatorial algorithms* (2011), 274–285.

[2] L. Lovász. Self-dual polytopes and the chromatic number of distance graphs on the sphere. *Acta Scientiarum Mathematicarum* 45:1–4 (1983), 317–323.

[3] A. Raigorodskii. On the chromatic numbers of spheres in $\mathbb{R}^n$. *Combinatorica* 32:1 (2012), 111–123.

Maps that take lines to plane curves

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(joint work with Vsevolod Petruschenko)

A *planarization* is a mapping f of an open subset U of the real projective plane into the real projective n-space, such that $f(L \cap U)$ is a subset of a hyperplane, for every line L. Studying planarizations is closely related to studying maps taking lines to curves of certain linear systems; a classical result of this type is the Möbius–von Staudt theorem [1, 4] about maps taking lines to lines, sometimes called the Fundamental Theorem of Projective Geometry. We assume that the planarizations are sufficiently smooth, i.e., sufficiently many times differentiable. We give [2, 3] a complete description of all planarizations in case $n = 3$ up to the following equivalence relation: two planarizations are equivalent if they coincide on a nonempty open set, after a projective transformation of the source space and a projective transformation of the target space. Apart from trivial cases, there are 16 equivalence classes, among which 6 classes of cubic rational maps (all remaining nontrivial classes are represented by quadratic rational maps).

The figures below illustrate the surfaces obtained as the images of the plane under the following 6 non-equivalent cubic rational planarizations:

(C_1): $[x:y:z] \mapsto [z(x^2+y^2) : y(x^2+z^2) : x(y^2+z^2) : xyz]$

(C_2): $[x:y:z] \mapsto [z(x^2-y^2) : y(x^2+z^2) : x(y^2-z^2) : xyz]$

(C_3): $[x:y:z] \mapsto [x^2z : z(x^2+y^2) : x(x^2+y^2-z^2) : y(x^2+y^2+z^2)]$

(C_4): $[x:y:z] \mapsto [x^2y : x(x^2-y^2) : z(x^2+y^2) : yz^2]$

(C_5): $[x:y:z] \mapsto [x^2(x+y) : y^2(x+y) : z^2(x-y) : xyz]$

(C_6): $[x:y:z] \mapsto [x^3 : xy^2 : 2xyz - y^3 : z(xz-y^2)]$.

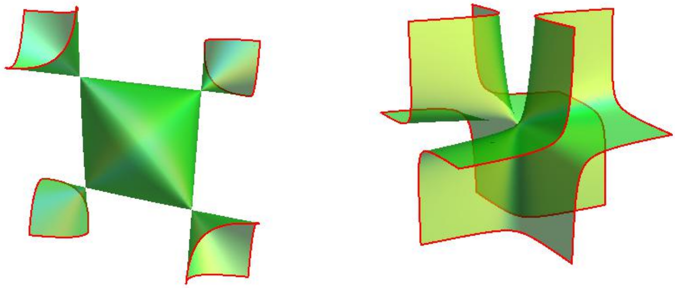

Figure 1: The surfaces parameterized by (C_1) (left) and by (C_2) (right).

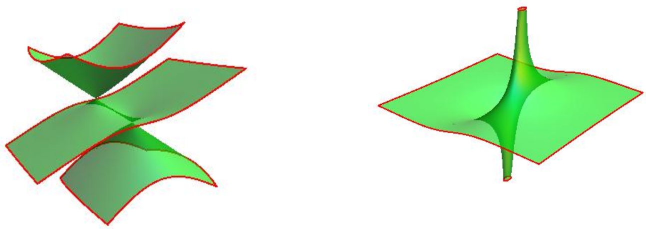

Figure 2: The surfaces parameterized by (C_3) (left) and by (C_4) (right).

[1] A. F. Möbius, *Der barycentrische Calcul*, 1827, In: August Ferdinand Möbius, gesammelte Werke, vol. 1, S. Hirzel (Ed.), Leipzig, 1885

[2] V. Petruschenko, V. Timorin. On maps taking lines to plane curves. arXiv:1409.3403

[3] V. Timorin. Planarizations and maps taking lines to linear webs of conics. Math. Research Letters **19** (2012), No. 4, 899–907.

[4] K. G. Ch. von Staudt, *Geometrie der Lage*, Nürnberg, 1847

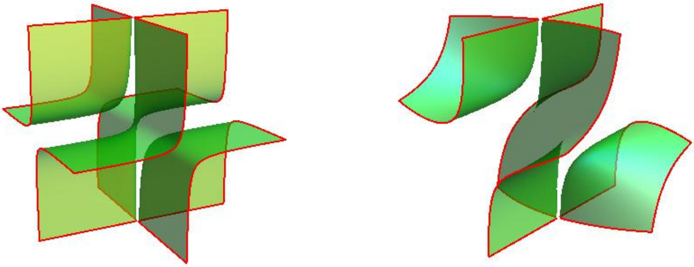

Figure 3: The surface parameterized by (C_5) (left) and by (C_6) (right).

Saturated 1-planar graphs

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(joint work with János Barát)

A graph is called 1-planar, if it can be drawn in the plane such that each edge is crossed at most once. It is known that the maximum number of edges of a 1-planar graph is $4n - 8$. Brandenburg et al. observed a very interesting phenomenon. They noticed that maximal 1-planar graphs (no edge can be added so that it remains 1-planar) can have much fewer edges.

I review the estimates of Brandenburg et al. for the minimum number of edges of a maximal 1-planar graph and give an improvement of the lower bound.

[1] F. J. Brandenburg, D. Eppstein, A. Gleissner, M. T. Goodrich, K. Hanauer, J. Reislhuber: On the Density of Maximal 1-Planar Graphs, Graph Drawing 2012, Lecture Notes in Computer Science **7704** (2013), 327-338.

[2] P. Eades, S.-H. Hong, N. Katoh, G. Liotta, P. Schweitzer, Y. Suzuki: Testing Maximal 1-Planarity of Graphs with a Rotation System in Linear Time, Graph Drawing 2012, Lecture Notes in Computer Science **7704** (2013), 339-345.

On tilings of the plane by polygons

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Are there polygons P that can be partitioned into two polygons so that both are similar to P? Here are three such examples:

- A "golden parallelogram" (any parallelogram whose width is $\sqrt{2}$ times bigger than its length); its median cuts it into two equal such parallelograms.
- Any right triangle; its altitude cuts it into two triangles that are similar to the original one.
- The Ammann's "Golden Bee" — a non-convex hexagon with right angles.

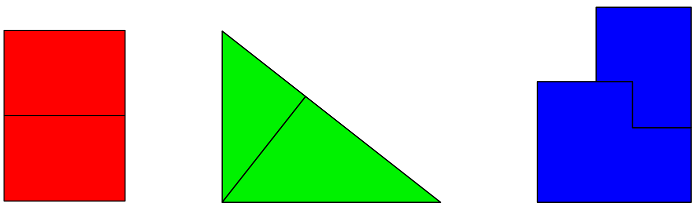

Figure 4: Substitutions.

Scherer [5] conjectured and Schmerl [4] proved that there are no other polygons with this property.

Each of these examples can be considered as a substitution: the original polygon is replaced by two its parts. We can then apply the same substitution to all large parts of the resulting tiling of the original polygon, then again and again. Tilings of the original polygons obtained in this way are called "supertiles".

Definition 1. A supertile is a tiling obtained from a single original tile by several substitutions from Fig. 4. Examples of supertiles are shown on Fig. 5.

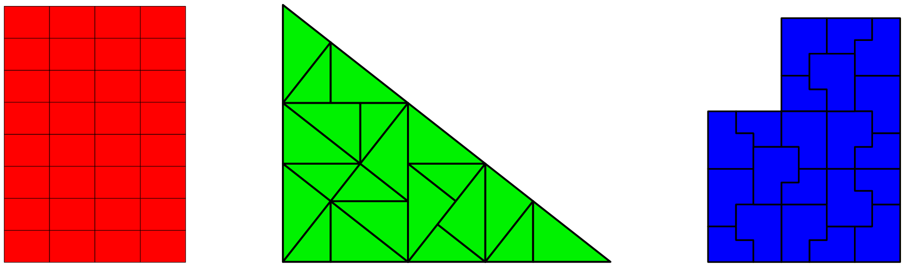

Figure 5: Supertiles for each of the three substitutions

Tilings of the plane that are "like" supertiles are called "self-similar".

Definition 2. ([6]) A *pattern* is a finite tiling. A pattern is *legal* if it is a subset of a supertile. A tiling T is called *self-similar* if all its finite subsets are legal.

Consider the following families of tilings of the plane:

- *Red tilings*: self-similar tilings by large and small golden rectangles.
- *Green tilings*: self-similar tilings by large and small golden right triangles.
- *Blue tilings* aka *Ammann A2 tilings*: self-similar tilings by large and small Golden Bees.

Theorem 7. *(a) There is a unique red tiling and that tiling is periodic. (b) There are continuum green tilings and all they are non-periodic [6]. (c) There are continuum blue tilings and all they are non-periodic [1].*

Definition 3. A family of tilings is called *SFT* if it can be defined by a finite number of local rules.

Theorem 8. *(a) The family of red tilings is SFT. (b) The family of green tilings is* **not** *SFT [7]. (c) The family of blue tilings is SFT [2].*

[1] R. Ammann, B. Grünbaum and G.C. Shephard. Aperiodic tiles. *Discrete and Computational Geometry,* v. 8 (1992). p. 1–25.

[2] Bruno Durand, Alexander Shen, Nikolay Vereshchagin. Ammann tilings: a classification and an application. arXiv:1112.2896 (2012)

[3] Branko Grunbaum, Geoffrey C. Shephard, Tilings and Patterns. Freeman, New York 1987.

[4] J. Schmerl. Dividing a polygon into two similar polygons. *Discrete Math.* v. 311 (2011), no. 4, p. 220–231.

[5] K. Scherer. A puzzling journey to the reptiles and related animals. Privately published, 1987.

[6] Boris Solomyak, Nonperiodicity implies unique composition for self-similar translationally finite tilings, Discrete and Computational Geometry 20 (1998) 265-279

[7] Nikolay Vereshchagin. Aperiodic Tilings by Right Triangles. In: Descriptional Complexity of Formal Systems - 16th International Workshop, DCFS 2014, Turku, Finland, August 5-8, 2014. Proceedings. Lecture notes in computer sciences, Vol. 8614. Berlin : Springer Verlag, 2014. P. 29-41.

Borsuk–Ulam type G-spaces

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(joint work with Oleg Musin)

We consider, for a finite group G, G-spaces that satisfy certain analogues of the Borsuk–Ulam theorem (BUT-spaces). In the case when G is an involution, there are several equivalent definitions for BUT-spaces that can be considered as properties. We are also going to discuss some combinatorial G-analogs of Tucker's lemma and other lemmas about G-spaces.

Self-affine convex bodies and bounded semigroups of affine operators

Andrey Voynov

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A convex body $K \subset \mathbb{R}^n$ is called self-affine if there is a set of nondegenerate affine operators $\{A_1, \dots, A_m\}$ such that:

- $K = \bigcup_{i=1}^{m} A_i K$, i.e. operators *divide* K;
- $\mathrm{int} A_i K \cap \mathrm{int} A_j K = \varnothing$, i.e. elements of partition may intersect each other by boundaries only.

Self-affine bodies appear in the theory of functional equations as domains of self-similar functions. The basic example of a self-affine body is a simplex divided by some triangulation into a certain number of simplices. Another example is a cylinder divided into two parts by contractions along its axis towards its bases.

We present a classification of self-affine bodies. It appears that, in some sense, all of them may be represented as a direct product of a certain self-affine polytope and an arbitrary convex body. We describe the relation between the geometry of a self-affine body K and the multiplicative semigroup generated by operators $\{A_1, \dots, A_m\}$ that divides it.

[1] Andrey Voynov. On the structure of self-affine convex bodies. *Mat. Sb.*, 2013, 204:8, 41–50.

[2] Vladimir Protasov, Andrey Voynov. Noncontractive compact semigroups of affine operators. *Mat. Sb.*, 2015, 206:7, 33–54.

[3] Vladimir Protasov, Andrey Voynov. Matrix semigroups with constant spectral radius. arXiv:1407.6568.

Symmetric multiple chessboard complexes and some theorems of Tverberg type

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(joint work with Duško Jojić and Rade Živaljević)

Generalizing the notion of chessboard complexes, we introduce the multiple chessboard complexes and the symmetric multiple chessboard complexes. We examine their topological properties and in some cases determine their connectivity and establish their shellability.

Using these results, we were able to establish some new Tverberg type, and colored Tverberg type theorems. One of them confirms a conjecture of Blagojević, Frick and Ziegler from [1] about the existence of "balanced Tverberg partitions".

The relevant publications on this topic are listed below.

[1] Pavle Blagojević, Florian Frick, Günter Ziegler. Tverberg plus constraints. Bull. London Math. Soc. 46 (2014), 953–967.

[2] Duško Jojić, Siniša Vrećica, Rade Živaljević. Multiple chessboard complexes and the colored Tverberg problem. arXiv:1412.0386

[3] Duško Jojić, Siniša Vrećica, Rade Živaljević. Symmetric multiple chessboard complexes and a new theorem of Tverberg type. arXiv:1502.05290

Discrete Morse theory for the moduli space of a flexible polygon, or Solitaire game on the circle

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(joint work with Gaiane Panina)

A *polygonal n-linkage* is a sequence of positive numbers $L = (l_1, \ldots, l_n)$. It should be interpreted as a collection of rigid bars of lengths l_i lying on a plane and joined consecutively in a cycle by revolving joints.

A subset I of $[n] = \{1, 2, ..., n\}$ is *short* if

$$\sum_I l_i < \frac{1}{2}\sum_1^n l_i.$$

A subset I of $[n] = \{1, 2, ..., n\}$ is *medium* if

$$\sum_I l_i = \frac{1}{2}\sum_1^n l_i.$$

A partition of $[n] = \{1, 2, ..., n\}$ into some subsets is called *admissible* if all the parts are short or there are exactly two medium sets.

The moduli space $M(L)$ of a linkage L is the set of all planar configurations of L modulo orientation-preserving isometries of $\mathbb{R}^2$. It is a well-studied mathematical object. M. Farber and D. Schütz [1] proved the following formula for its Betti numbers:

$$\beta_k = a_k + a_{n-3-k} + b_k,$$

where a_k is the number of short $(k+1)$-subsets of $[n]$ containing the longest bar, and β_k is the number of medium $(k+1)$-subsets of $[n]$ containing the longest bar.

We consider only generic linkages, i. e. the linkages with no medium subsets. We have a structure of a regular CW-complex $\mathcal{K}(L)$ on the moduli space $M(L)$ [3]. The k-dimensional cells of this complex are labeled by cyclically ordered admissible partitions of the set $[n]$ into $(n-k)$ non-empty subsets, and a closed cell C belongs to the boundary of some other closed cell C' if and only if the partition $\lambda(C)$ is finer than $\lambda(C')$ (see Fig. 1 for an example).

The number of cells exceeds the sum of Betti numbers. We use Robin Forman's discrete Morse theory [2] to reduce the number of cells. It is a very powerful technique (at least as powerful as the smooth Morse theory is): it allows to compute homology, cup-product, Novikov's homology, develop Witten's deformation of the Laplacian, etc. In this talk we demonstrate how it works: We build a perfect discrete Morse function on $\mathcal{K}(L)$. Should be mentioned that not all the manifolds possess a perfect Morse function. Even if it is the case, it is difficult to find it. In particular, in the discrete setting it is an NP-hard problem.

The discrete perfect Morse function is constructed in two steps. On the first step, we introduce some natural pairing on the cell complex which substantially reduces the number of critical cells. However, this number is not yet minimal possible. The rules of manipulating with the

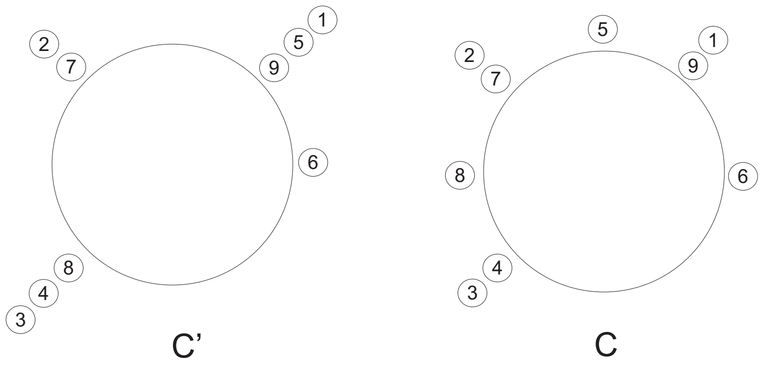

Figure 6: The cell C belongs to the boundary of the cell C'

cells, and the rules describing gradient paths resemble the solitaire game. On the second step we (following once again R. Forman) apply path reversing technique, which gives a perfect Morse function. This technique is the discrete version of Milnor's "First Cancellation Theorem" [3].

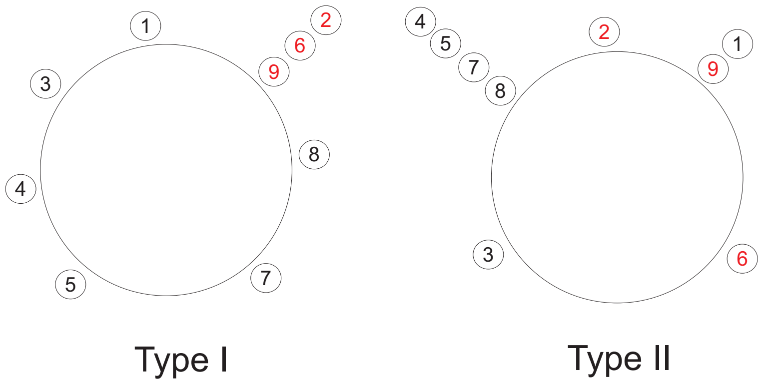

Figure 7: Cells of two types corresponding to the subset 2,6,9

We divide the cells that survived the contracting into two types and give the bijection between the short subsets of $[n]$ containing n and the pairs of the cells (one cell of each type for every subset). For example, if $n = 9$ and $L = (1, 1, 1, 1, 1, 1, 1, 1, 1)$, then the subset $\{2, 6, 9\}$ corresponds to the cells given on Fig. 2 of dimension 2 (type I) and 4 (type II).

[1] M. Farber, D. Schütz. Homology of planar polygon spaces. *Geom. Dedicata* 125 (2007), 75–92.

[2] R. Forman. A user's guide to discrete Morse theory. *Sem. Lothar. Combin.* 48B:48c (2002).

[3] G. Panina. Moduli space of a planar polygonal linkage: a combinatorial description. (2012) arXiv:1209.3241.

[4] G. Panina, A. Zhukova. Discrete Morse theory for moduli spaces of flexible polygons, or solitaire game on the circle. (2015) arXiv:1504.05139.

Equivariant Methods in Discrete Geometry: Problems and Progress

Günter Ziegler

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(joint work with Pavle Blagojević, Florian Frick, Albert Haase, and Benjamin Matschke)

In this lecture, I will discuss three different problems from Discrete Geometry,

- the Topological Tverberg Problem,
- the Colored Tverberg Problem, and
- the Grünbaum Hyperplane Problem.

These problems have many things in common:

- they are easy to state, and may look harmless,
- they have very nice and classical configuration spaces,
- they may be attacked by “Equivariant Obstruction Theory”,
- this solves the problems — but only partially,
- which leads us to ask more questions, look for new tools…
- and this yields surprising new results.

The hyperplane measure equipartition problem revisited

Rade Živaljević

Mathematical Institute SASA, Belgrade

We give an overview and the history of the last 20 years of the hyperplane measure equipartition problem, including some critical comments on the review paper:

[1] Pavle V. M. Blagojević, Florian Frick, Albert Haase, Günter M. Ziegler. Topology of the Grünbaum-Hadwiger-Ramos hyperplane mass partition problem. arXiv:1502.02975 [math.AT] (2015).