Conjectures in combinatorial and convex geometry

This note collects several conjectures, formulated in my papers.

(1) *Let the plane $\mathbb{R}^2$ be partitioned into closed convex sets $V_1, \dots, V_m$. Let $v_1, \dots, v_m$ be a set of vectors. Then there exists a permutation $\sigma \in \mathfrak{S}_m$ such that the sets*

$$V_i' = V_i + v_{\sigma(i)}$$

cover $\mathbb{R}^2$. There also exists a permutation $\tau \in \mathfrak{S}_m$ such that the sets

$$V_i' = V_i + v_{\tau(i)}$$

do not overlap pairwise (their interiors are pairwise disjoint).

This conjecture was formulated in [4], and a particular case was proved for the so called *hierarchically affine* partitions.

(2) (The generalized Grünbaum conjecture) *Consider a family $\mathcal{F}$ of translates of a convex compact set K in $\mathbb{R}^d$. Suppose that any d or less sets in $\mathcal{F}$ have a common point. Then there exists a set T of $d+1$ points that intersects any set in the family $\mathcal{F}$.*

This conjecture was proved in [3] for the case $d = 2$, the conjecture is formulated in [5].

(3) (The dual Tverberg theorem) *Suppose we are given a family $\mathcal{F}$ of $(d+1)n$ hyperplanes in general position in $\mathbb{R}^d$. Then $\mathcal{F}$ can be partitioned into n sets of $d+1$ hyperplanes each so that the n simplexes, bounded by each set of $d+1$ hyperplanes, have a common point.*

This conjecture was proved in [6] for the case when n is a prime power. The conjecture was also proved for $d = 2$ and arbitrary n, in this case it follows from the central point theorem.

(4) (The colored dual Tverberg theorem) We have the following problem, that could be a natural analogue of the colorful Tverberg theorem [1, 7, 2].

Find least possible $t = t(d, r)$ such that the following holds. Suppose $(d+1)t$ hyperplanes in general position are given in $\mathbb{R}^d$, and they are colored into $d+1$ colors, so that each color is used t times. Then we can select r disjoint rainbow *$(d+1)$-tuples of hyperplanes so that r simplexes, bounded by the corresponding $(d+1)$-tuples, have a common point.*

In was conjectured in [6] that $t(2, r) = r$, but a counterexample to this conjecture was found by Li Ping (a student of Imre Bárány) in the case $r = 2$. The results of [7] and [2] imply that for r a prime power $t(d, r-d) \leq 2r$ and for r prime $t(d, r-d-1) \leq r$.

Thus, even the plane case ($d = 2$) of this problem remains to be interesting.

Список литературы

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